BQP After 29 Years

An unapologetically retro topic!

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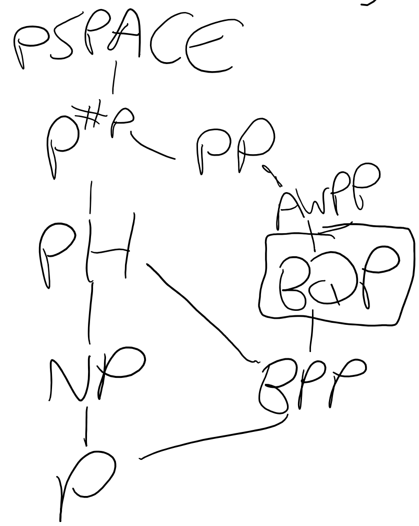
BQP: Bounded-Error Quantum Polynomial-Time
(Bernstein & Vazirani, 1993)



BBBV'94: $BQP^{BQP} = BQP$

Profound Questions That Have Remained Open:

1. $BPP = BQP$?
2. $NP \subseteq BQP$?
3. $BQP \subseteq NP$? AM? PH?

And many more arcane ones, e.g.

$$NP^{BQP} \subseteq BQP^{NP} ?$$

(Fortnow 2005)

Formally, we can do little better than point out some simple relationships among these questions.

E.g., $NP \subseteq BQP$ AND $BQP \subseteq AM$

$\Rightarrow PH$ collapses

(since then $coNP \subseteq AM$)

For BPP, we can say much more

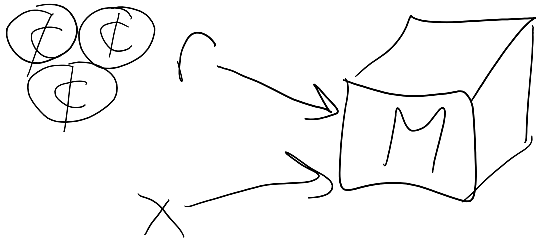
than for BQP. E.g.,

- $BPP \subseteq PH$ (Sipser–Gács–Lautemann)

  So, $P=NP \Rightarrow P=BPP$

- $BPP \subset P/poly$ (Adleman)
- $NP \subseteq BPP \Rightarrow PH \subseteq BPP$ & PH collapses

All these results depend on "pulling out the randomness" of a classical randomized algorithm M



$\Pr_r[M(x,r) \text{ accepts}]$

Nothing analogous seems to be possible for quantum algorithms! ☹

This difference is crucial in, e.g., the analysis of sampling-based quantum supremacy experiments (A.–Arkhipov, Bremner–Jozsa–Shepherd...)

But how could we ever prove these differences between BPP and BQP?

Like perturbation theory for physicists, relativization is complexity theorists' way to prove things even when it's too hard to prove things.

Bernstein-Vazirani '93, Simon '94, Shor '94: There exist oracles A relative to which $BPP^A \neq BQP^A$.

BBBV '94: There's an oracle A such that $NP^A \not\subset BQP^A$
(I.e., Grover is optimal; any poly-time quantum algorithm for NP-complete problems must be non-black-box)

Fortnow-Rogers '98: There's an oracle relative to which $P = BQP$ yet

PH is infinite

(A.–Chen 2017: Even relative to which quantum and classical polytime approximate sampling coincide, yet PH is infinite)

Bernstein-Vazirani '93, Watrous '2000:
There are oracles relative to which $BQP \not\subset NP$ and even $BQP \not\subset MA$

But what about $BQP \not\subset AM$?
$BQP \not\subset PH$?

These oracle separations were huge open problems for 25 years!

A. 2009: FORRELATION, a candidate problem for these separations.

Given black-box functions
$$f, g: \{0,1\}^n \to \{-1,1\}.$$

Promised that either
(i) They're uniformly random & independent, or
(ii) They're uniformly random individually, but g is correlated with $\hat{f}$ (f's Boolean Fourier transform).
Decide which.

"forrelated"

I showed how to solve FORRELATION with only 1 (!) quantum query to f, g.

I conjectured it was not in $PH^{f,g}$.

Raz-Tal 2018 finally proved my conjecture, with a breakthrough analysis of the Fourier spectra of AC^0 circuits.

$\Rightarrow$ $\boxed{\exists \text{ oracle relative to which } BQP \not\subset PH}$

Indeed, they proved something stronger: a PH machine can guess whether f, g are uniform or forrelated with bias at most $\frac{1}{\exp(n)}$.

Now I'd like to share some recent work with William Kretschmer & DeVon Ingram (arXiv:2111.10409)

Our first question was which <u>other</u> longstanding open problems about BQP relative to oracles could be knocked off using Raz & Tal's breakthrough

<u>Easy</u>: There exist oracles relative to which

- $BQP \not\subset NP/poly$
- $P = NP \neq BQP$

<u>Idea</u>: First use FORRELATION to make $P \neq BQP$. Then, $\forall k \geq 1$, recursively

encode into the oracle A the answers to all possible NP^A queries, using strings of length $\sim n^k$. This makes $P^A = NP^A = PH^A$, without re-collapsing P and BQP (by Raz-Tal).

More interesting:

There exists an oracle relative to which
$$P = NP \neq BQP = P^{\#P}.$$

(BQP: "I will not let my aspirations be constrained by NP's, unlike my weaker cousin BPP!")

Idea: First use, e.g., exp(n)-sized MAJORITY instances to make $PH \neq P^{\#P}$. Then, $\forall k \geq 1$, and using strings of size $\sim n^k$, recursively:

- Encode into the oracle A the answers to all possible NP^A queries

- Encode into A the answers to all possible $\#P^A$ queries, <u>but</u> hide them in FORRELATION instances.

This way, we make $P^A = NP^A$ and $BQP^A = P^{\#P^A}$ but by Raz-Tal + a hybrid argument, we don't recollapse PH^A & $P^{\#P^A}$.

<u>Another new result:</u>

There exists an oracle relative to which $NP \subseteq BQP$ but PH is infinite.

("If a fast quantum algorithm for NP-complete problems collapses PH, it can only be for a non-relativizing reason.")

<u>Idea:</u> First, use a random oracle — this makes PH infinite with probability 1, by Rossman-Servedio-Tan 2015.

Then, encode into the oracle A the answers to all possible NP^A queries, <u>but</u> hidden in FORRELATION instances. This makes $NP^A \subseteq BQP^A$. But by Raz-Tal + hybrid argument, it still looks uniformly random to PH^A, so by Rossman-Servedio-Tan it doesn't recollapse PH.

To go further, we needed more than Raz-Tal. Since the 80s, a central tool for oracle separations (e.g. $PH \neq PSPACE$) has been <u>random restrictions</u>.

Our <u>Random Restriction Lemma for BQP</u>

Suppose quantum alg. Q makes T queries to $x \in \{0,1\}^N$. Let $0 \leq k \leq 2pN$. If we sample $S \subseteq [N]$ s.t. each $i \in [N]$ is in S w.p. p, then w.p. $\geq 1 - e^{-k/6}$ over S, $\exists L \subseteq S$ with $|L| \leq k$ s.t. for every $y \in \{0,1\}^N$ that differs from x only on $S \setminus L$,

$$|\Pr[Q(x) \text{ accepts}] - \Pr[Q(y) \text{ accepts}]| \leq 16Tp\sqrt{\frac{N}{k}}.$$

<u>Proof Idea</u>: Just careful BBBV!

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Example Application: (though not yet using its full power)

$\forall k$, there's an oracle relative to which

$$\Sigma_{k+1}^{P} \not\subset BQP^{\Sigma_k^P}$$

(Indeed a random oracle!)

Proof Idea: Just plug our random restriction lemma into Rossman-Servedio-Tan's proof that PH is infinite relative to a random oracle.

Conjecture:
$\forall k$, there's an oracle relative to which

$$\Sigma_k^P \subseteq BQP \text{ but } \Sigma_{k+1}^P \not\subset BQP.$$

NOTE 1: Suppose $NP^{BQP} \subseteq BQP^{NP}$.

Then $NP \subseteq BQP \Rightarrow$

$$NP^{NP} \subseteq NP^{BQP} \subseteq BQP^{NP} \subseteq BQP^{BQP} = BQP.$$

Hence, in giving an oracle world where $NP \subseteq BQP$ but $\Sigma_2^P \not\subset BQP$, we've also given a world where $NP^{BQP} \not\subset BQP^{NP}$, thereby solving Fortnow's problem from 2005!

NOTE: All these oracle results are only possible in a post-Raz-Tal world— for if $BQP \subseteq AM$, then the statements would all be false, via 'relativizing proofs'!

Another application of our random restriction lemma

There's an oracle relative to which

$$PP \not\subset BQP^{NP}$$

.... & even $BQP^{NP^{BQP^{NP^{\cdots}}}}$

.... & even $QMA^{QMA^{QMA^{\cdots}}}$

.... & even $(MIP^*)^{(MIP^*)^{(MIP^*)^{\cdots}}}$

(yes, even though in the "real world," $MIP^* = RE$!!)

There's an oracle relative to which

$$P = NP = BQP = QMA \neq PP$$

Contrast PostBQP = PP with PostBPP, which is relativizingly in BPP^{NP}!

Summary: Relative to suitable oracles, we

Summary Relative to suitable oracles, we can almost completely "unshackle" BQP from the complexity classes around it, by exploiting QCs' ability to solve FORRELATION combined with their inability to beat Grover. Even while BPP remains shackled!

Some Problems We Didn't Solve

- Oracle where P=NP but BQP=EXP?
  BQP=NEXP?
- Oracle where $QMA \not\subset BQP^{NP}$?
- Sharper random restriction lemma for BQP & QMA?

OK, but what about the unrelativized world???

My bets/speculations:

- $P = BPP \neq BQP$
- $NP \not\subset BQP$

No idea

- $NP \not\subset BQP$
- $BQP \subseteq NP$?? ← No idea on this one!
    - Under derandomization, $BQP \subseteq AM$ would suffice.
    - Is there an explicit instantiation of FORRELATION?
    - Is there a decision version of BOSON SAMPLING?

Thanks for listening!