Outline

1. The Adiabatic Theorem
2. The Adiabatic Algorithm
3. Computational Complexity
4. Notes on Adiabatic Universality
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The Adiabatic Theorem

Hamiltonian: A hermitian operator with eigenvalues describing the energy eigenstates of the system.

A quantum system with a time-changing Hamiltonian will stay in the same energy level if the rate-of-change is slow enough.

\[ T \gg \frac{2\pi \hbar}{\Delta} \]

(with level separation \( \Delta \)).

Consider:

\[
H_i = \begin{pmatrix} E_0 & \delta \\ \delta & E_1 \end{pmatrix}, \quad H_T = \begin{pmatrix} E_1 & \delta \\ \delta & E_0 \end{pmatrix}
\]

\[
H(s) = (1 - s)H_i + sH_T
\]
For simplicity, take $E_0 = 0$, $E_1 = 1$. Plot eigenvalues as a function of $s \in [0, 1]$

\[
\begin{align*}
\delta &= 0 \\
\text{With } s \text{ varied over time } T &\gg \frac{2\pi \hbar}{\Delta}, \text{ system will remain in the same level.}
\end{align*}
\]
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The Adiabatic Algorithm

- Encode problem as SAT\(^1\)
- Each clause has a local Hamiltonian \(H_c\) encoding the assignment of variables, \(H_T = \sum H_c\)
- Initialize system into simple ground state of some Hamiltonian \(H_0\).
- Adiabatically evolve Hamiltonian to \(H_T\): system will be in ground state encoding solution

The Adiabatic Algorithm

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\[H(0) = H_0, \ H(T) = H_T\]
\[\rightarrow H(t/T = s) = (1 - s)H_0 + sH_T\]
- Vary \(s\) slowly enough such that system remains in ground state

Schroedinger Equation

\[ i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle \]

Introduce a Delay Factor \( \tau(s) \) indicating how slowly the Hamiltonian varies\(^2\).

\[ \frac{d}{ds} |\psi(s)\rangle = -i \tau(s) H(s) |\psi(s)\rangle \]

Adiabatic evolution requires:

\[ \tau(s) \gg \| \frac{d}{ds} H(s) \|_2 \frac{g(s)^2}{g^2} \]

Evolution time \( T \) proportional to separation \( g_{min}^{-2} \).

\(^2\)Van Dam, Mosca, Vazirani, 2008.
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Adiabatic Complexity

Does not in general indicate 3-SAT soluble in polynomial time: some problems have $g_{\text{min}}$ exponentially small

- But can recover Grover Search$^3$
- Consider

$$f(x) : \{0, 1\}^n \rightarrow \mathbb{R} = \begin{cases} 0 & \text{if } x \text{ is the solution} \\ 1 & \text{otherwise} \end{cases}$$

Final Hamiltonian $H_x = \sum_{z \in \{0, 1\}^n \setminus \{x\}} |z\rangle \langle z|$
Initial Hamiltonian with Hadamard $|+\rangle^n = |\hat{0}\rangle^n$ as ground state.

$$H_0 = \sum_{z \in \{0,1\}^n \setminus \{0^n\}} |\hat{z}\rangle \langle \hat{z}|$$

Level separation of $H(s) = (1 - s)H_0 + sH_x$

$$g(s) = \sqrt{N + 4(N - 1)(s^2 - s)}$$

Looks like $T \propto g_{\min}^{-2} = O(N)$. But if we let the delay vary in time:\(^4\)

$$T = \int_{s=0}^{1} \frac{ds}{g(s)^2} = \frac{N \cdot \text{arctan}(\sqrt{N-1})}{\sqrt{N-1}} = O(\sqrt{N})$$

\(^4\text{Van Dam, Mosca, Vazirani, 2008.}\)
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Can show that we can polynomially simulate a general Quantum Circuit adiabatically

- Quantum Circuit: L gates, state after gate \( \ell \) is \( |\alpha(\ell)\rangle \)
- Adiabatically, could have final Hamiltonian \( H_L \) with g.s. \( |\alpha(L)\rangle \)

Problems: Can’t always specify \( H_L \) without knowing \( |\alpha(L)\rangle \)
Kitaev’s history state

\[ |\eta\rangle = \frac{1}{\sqrt{L+1}} \sum_{\ell=0}^{L} |\alpha(\ell)\rangle \otimes |1^\ell 0^{L-\ell}\rangle \]

- Define final Hamiltonian \( H_f \) to have \( |\eta\rangle \) as ground state.\(^5\)
- Initial Hamiltonian \( H_0 \) has g.s. \( |\alpha(0)\rangle \otimes |0^L\rangle \)
- Can construct \( H_0 \) and \( H_f \) without knowing \( |\alpha(L)\rangle \)
- Measure: If clock is \( |1^\ell\rangle \) then other register carries result
- Can simulate a QC using 5-local Hamiltonians in \( O(L^5) \)

Can generalize to 3-local H, or a grid with two-local H and six-state particles.

Conclusions

- Can implement quantum computations by adiabatic evolution
- Slowness of evolution related to complexity of the problem (not known in general)
- Recover $O(\sqrt{N})$ of Grover
- Can implement any quantum circuit adiabatically