Influences in low-degree polynomials

Artūrs Bačkurs

Let $p : \mathbb{R}^n \to \mathbb{R}$.

Definition

$$\mathrm{Var}(p) := \mathrm{E}_{x,y \in \{-1,1\}^n} \left[(p(x) - p(y))^2 \right].$$

Definition

The influence of the i^{th} variable is

$$\mathrm{Inf}_i(p) := \mathrm{E}_{x \in \{-1,1\}^n} \left[(p(x) - p(x^i))^2 \right],$$

where x^i is x with the i^{th} bit flipped.

Aaronson-Ambainis conjecture

Aaronson-Ambainis conjecture:

Conjecture

Suppose that $p : \mathbb{R}^n \to \mathbb{R}$ *is degree-*d *polynomial and* $|p(x)| \leq 1$ *for all* $x \in \{-1, 1\}^n$*. Then there exists an* $i \in [n]$ *such that* $\mathrm{Inf}_i(p) \geq (\mathrm{Var}(p)/d)^{O(1)}$*.*

Boolean case

Theorem

For every $p : \{-1,1\}^n \to \{-1,1\}$ *we have*

$$\mathrm{Inf}_{\max}(p) \geq \frac{\mathrm{Var}(p)}{D(p)},$$

where $D(p)$ *is the deterministic decision tree complexity of* p*.*

Using $D(p) \leq O(\deg(p)^4)$ we get

$$\mathrm{Inf}_{\max}(p) \geq \Omega(\mathrm{Var}(p)/\deg(p)^4).$$

Case of symmetric polynomial

Theorem

Let $p : \{-1,1\}^n \to \{-1,1\}$ *be a symmetric polynomial. For all* i*,*

$$\mathrm{Inf}_i(p) = \Omega\left(\frac{\mathrm{Var}(p)^3}{\deg^4(p)\ln(1/\mathrm{Var}(p))}\right).$$

Exponential lower bound

Definition

A function $p : \{-1,1\}^n \to \mathbb{R}$ *is called an* (δ, j)*-junta if there exists a function* $g : \{-1,1\}^n \to \mathbb{R}$ *depending on at most* j *coordinates such that* $\mathrm{E}_{x\in\{0,1\}^n}[(p(x)-g(x))^2] \le \delta$.

Theorem

Let $p : \{-1,1\}^n \to [-1,1]$, $k \ge 1$, *and* $\delta > 0$. *Suppose*

$$\sum_{|S|>k} \widehat{p}(S)^2 \le \exp(-O(k^2 \log k)/\delta).$$

Then p *is an* $(\delta, 2^{O(k)}/\delta^2)$*-junta.*

Exponential lower bound

An exponential version of Aaronson-Ambainis conjecture holds:

Theorem

Suppose that p is degree-d polynomial and $|p(x)| \leq 1$ for all $x \in \{-1, 1\}^n$. Then there exists an $i \in [n]$ such that $\mathrm{Inf}_i(p) \geq (\mathrm{Var}(p)/2^d)^{O(1)}$.

Consequences of Aaronson-Ambainis conjecture

Theorem

Suppose Aaronson-Ambainis conjecture holds.
Let Q be a quantum algorithm that makes T queries to a Boolean input $X = (x_1, ..., x_N)$, and let $\epsilon > 0$. Then there exists a deterministic classical algorithm that makes $\text{poly}(T, 1/\epsilon, 1/\delta)$ *queries to the x_i's, and that approximates Q's acceptance probability to within an additive arror ϵ on a $1 - \delta$ fraction of inputs.*

Consequences of Aaronson-Ambainis conjecture

Definition

Let $\mathsf{D}_\epsilon(f)$ *be the minimum number of queries made by a deterministic algorithm that evaluates* f *on at least* $1-\epsilon$ *fraction of inputs. Similarly define* $\mathsf{Q}_\epsilon(f)$.

Theorem

Suppose Aaronson-Ambainis conjecture holds.
Then $\mathsf{D}_{\epsilon+\delta}(f) \le (\mathsf{Q}_\epsilon(f)/\delta)^{O(1)}$ *for all Boolean functions* f *and all* $\epsilon, \delta > 0$.

Consequences of Aaronson-Ambainis conjecture

Definition

AvgP *is the class of languages for which there exists a polynomial-time algorithm that solves a* $1 - o(1)$ *fraction of instances.*

Theorem

Suppose Aaronson-Ambainis conjecture holds.
Then $\mathsf{P} = \mathsf{P}^{\#\mathsf{P}}$ *implies* $\mathsf{BQP}^A \subset \mathsf{AvgP}^A$ *with probability* 1 *for a random oracle* A.

Consequences of Aaronson-Ambainis conjecture

Unconditional results:

Theorem

Suppose a quantum algorithm makes T queries to a Boolean input $x \in \{0,1\}^n$. Then for all $\alpha, \delta > 0$, we can approximate the acceptance probability to within an additive constant α, on a $1-\delta$ fraction of inputs, by making $\frac{2^{O(T)}}{\alpha^4 \delta^4}$ deterministic classical queries.

Theorem

$\mathrm{D}_{\epsilon+\delta}(f) \leq 2^{O(Q_\epsilon(f))}/\delta^4$ *for all Boolean functions f and all $\epsilon, \delta > 0$.*

Thank you!