Influences in low-degree polynomials

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Let $p : \mathbb{R}^n \to \mathbb{R}$.

**Definition**

\[ \text{Var}(p) := E_{x,y \in \{-1,1\}^n} \left[ (p(x) - p(y))^2 \right]. \]

**Definition**

The influence of the $i^{\text{th}}$ variable is

\[ \text{Inf}_i(p) := E_{x \in \{-1,1\}^n} \left[ (p(x) - p(x^i))^2 \right], \]

where $x^i$ is $x$ with the $i^{\text{th}}$ bit flipped.
Aaronson-Ambainis conjecture:

Suppose that $p : \mathbb{R}^n \to \mathbb{R}$ is degree-$d$ polynomial and $|p(x)| \leq 1$ for all $x \in \{-1, 1\}^n$. Then there exists an $i \in [n]$ such that $\operatorname{Inf}_i(p) \geq (\operatorname{Var}(p)/d)^{O(1)}$. 
Boolean case

Theorem

For every $p : \{-1, 1\}^n \rightarrow \{-1, 1\}$ we have

$$\inf \max (p) \geq \frac{\text{Var}(p)}{D(p)},$$

where $D(p)$ is the deterministic decision tree complexity of $p$.

Using $D(p) \leq O(\text{deg}(p)^4)$ we get

$$\inf \max (p) \geq \Omega(\text{Var}(p)/\text{deg}(p)^4).$$
Theorem

Let $p : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be a symmetric polynomial. For all $i$,

$$\inf_i(p) = \Omega \left( \frac{\text{Var}(p)^3}{\text{deg}^4(p) \ln(1/\text{Var}(p))} \right).$$
Definition

A function $p : \{-1, 1\}^n \to \mathbb{R}$ is called an $(\delta, j)$-junta if there exists a function $g : \{-1, 1\}^n \to \mathbb{R}$ depending on at most $j$ coordinates such that $E_{x \in \{0,1\}^n}[(p(x) - g(x))^2] \leq \delta$.

Theorem

Let $p : \{-1, 1\}^n \to [-1, 1]$, $k \geq 1$, and $\delta > 0$. Suppose

$$\sum_{|S| > k} \hat{p}(S)^2 \leq \exp(-O(k^2 \log k)/\delta).$$

Then $p$ is an $(\delta, 2^{O(k)}/\delta^2)$-junta.
Exponential lower bound

An exponential version of Aaronson-Ambainis conjecture holds:

**Theorem**

*Suppose that $p$ is degree-$d$ polynomial and $|p(x)| \leq 1$ for all $x \in \{-1, 1\}^n$. Then there exists an $i \in [n]$ such that $\inf_i (p) \geq (\text{Var}(p)/2^d)^{O(1)}$.**
Consequences of Aaronson-Ambainis conjecture

Theorem

Suppose Aaronson-Ambainis conjecture holds. Let $Q$ be a quantum algorithm that makes $T$ queries to a Boolean input $X = (x_1, ..., x_N)$, and let $\epsilon > 0$. Then there exists a deterministic classical algorithm that makes $\text{poly}(T, 1/\epsilon, 1/\delta)$ queries to the $x_i$’s, and that approximates $Q$’s acceptance probability to within an additive error $\epsilon$ on a $1 - \delta$ fraction of inputs.
Consequences of Aaronson-Ambainis conjecture

**Definition**

Let $D_\epsilon(f)$ be the minimum number of queries made by a deterministic algorithm that evaluates $f$ on at least $1 - \epsilon$ fraction of inputs. Similarly define $Q_\epsilon(f)$.

**Theorem**

Suppose Aaronson-Ambainis conjecture holds. Then $D_{\epsilon+\delta}(f) \leq (Q_\epsilon(f)/\delta)^{O(1)}$ for all Boolean functions $f$ and all $\epsilon, \delta > 0$. 
Definition

\( \text{AvgP} \) is the class of languages for which there exists a polynomial-time algorithm that solves a \( 1 - o(1) \) fraction of instances.

Theorem

Suppose Aaronson-Ambainis conjecture holds. Then \( P = \mathbb{P} \) implies \( \text{BQP}^A \subset \text{AvgP}^A \) with probability 1 for a random oracle \( A \).
### Unconditional results:

**Theorem**

Suppose a quantum algorithm makes $T$ queries to a Boolean input $x \in \{0, 1\}^n$. Then for all $\alpha, \delta > 0$, we can approximate the acceptance probability to within an additive constant $\alpha$, on a $1 - \delta$ fraction of inputs, by making $\frac{2^{O(T)}}{\alpha^4 \delta^4}$ deterministic classical queries.

**Theorem**

$D_{\epsilon+\delta}(f) \leq 2^{O(Q_{\epsilon}(f))}/\delta^4$ for all Boolean functions $f$ and all $\epsilon, \delta > 0$. 
Thank you!