CS395T Problem Set 1: Quantum Basics

1. **Stochastic and unitary matrices.**

   (a) Show that a matrix $A \in \mathbb{R}^{n \times n}$ maps every nonnegative vector $v \in \mathbb{R}^n_{\geq 0}$ to a nonnegative vector Av with the same 1-norm, if and only if A is *stochastic* (that is, A is a nonnegative matrix all of whose columns sum to 1).

   (b) Show that a matrix $U \in \mathbb{C}^{n \times n}$ maps every vector $v \in \mathbb{C}^n$ to a vector Uv with the same 2-norm, if and only if U is unitary (that is, $UU^\dagger = I$, where $U^\dagger$ denotes the conjugate transpose of U).

2. **Perfectly-distinguishable quantum states.** Show that there exists a measurement to distinguish the quantum states $|\psi\rangle$ and $|\varphi\rangle$ with certainty, if and only if $\langle\psi|\varphi\rangle = 0$.

3. **CHSH/Bell inequality.** Consider the following game played by Alice and Bob. Alice receives a bit x and Bob receives a bit y, with both bits uniformly random and independent. The players win if Alice outputs a bit a and Bob outputs a bit b such that $a + b = xy \pmod 2$. (Alice and Bob are cooperating in this game, not competing.) The players can agree on a strategy in advance, but once they receive x and y no further communication between them is allowed.

   (a) Give a deterministic strategy by which Alice and Bob can win this game with 3/4 probability.

   (b) Show that no deterministic strategy lets them win with more than 3/4 probability.

   (c) Show that no probabilistic strategy lets them win with more than 3/4 probability.

   Now suppose Alice and Bob share the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, with Alice holding one qubit and Bob holding the other qubit. Suppose they use the following strategy: if $x = 1$, then Alice rotates her qubit by $\pi/4$ clockwise in the $\{|0\rangle, |1\rangle\}$ plane; otherwise she does nothing. She then measures her qubit in the $\{|0\rangle, |1\rangle\}$ basis and outputs the result. If $y = 1$, then Bob rotates his qubit by $\pi/8$ counterclockwise in the $\{|0\rangle, |1\rangle\}$ plane; otherwise he rotates by $\pi/8$ clockwise. He then measures his qubit in the standard basis and outputs the result.

   d. Show that, regardless of x and y, Alice and Bob win the game with probability $\cos^2 \frac{\pi}{8} = \frac{1+\sqrt{1/2}}{2}$ using this strategy. Conclude that Alice and Bob win with greater overall probability than would be possible in a classical universe.

   You've just proved the *CHSH/Bell Inequality*—one of the most famous results of quantum mechanics—which showed the impossibility of Einstein's dream of removing "spooky action at a distance" from quantum mechanics. Alice and Bob's ability to win the above game more than 3/4 of the time using quantum entanglement was experimentally confirmed in the 1980's.

4. **GHZ paradox.** Consider the following game: Alice, Bob, and Charlie are given input bits a, b, and c respectively. They are promised that $a \oplus b \oplus c = 0$ (where $\oplus$ denotes addition mod 2). Their goal is to output bits x, y, and z respectively such that $x \oplus y \oplus z = a \vee b \vee c$. They can agree on a strategy in advance but cannot communicate after receiving their inputs.

   (a) Show that in a classical universe, there is no strategy that enables them to win this game with certainty.

(b) Suppose Alice, Bob, and Charlie share the entangled state

$$\frac{1}{2}\left(|000\rangle - |011\rangle - |101\rangle - |110\rangle\right).$$

Show that *now* there exists a strategy by which they can win the game with certainty. [*Hint:* Have each player measure its qubit in one basis if its input bit is 0, or in a different basis if its input bit is 1.]

5. **Density matrices.** Suppose we don't know which quantum state we have, and instead just have a probability distribution in which the state $|\psi_i\rangle = \alpha_{i1}|1\rangle + \cdots + \alpha_{in}|n\rangle$ occurs with probability p_i, for all $i \in \{1, \ldots, m\}$. (Here $p_1 + \cdots + p_m = 1$.) Consider the $n \times n$ matrix

$$\rho = \sum_{i=1}^{m} p_i |\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{m} p_i \begin{pmatrix} \alpha_{i1}\alpha_{i1}^* & \cdots & \alpha_{i1}\alpha_{in}^* \\ \vdots & \ddots & \vdots \\ \alpha_{in}\alpha_{i1}^* & \cdots & \alpha_{in}\alpha_{in}^* \end{pmatrix}$$

(where * denotes complex conjugate). We call ρ the *density matrix* corresponding to the *ensemble* $\{|\psi_i\rangle, p_i\}$.

(a) Show, as a consequence of the Born rule applied to pure states, that if we measure a quantum system with density matrix ρ in the standard basis, we get outcome $|i\rangle$ with probability $(\rho)_{ii}$ (i.e., the (i,i) entry of ρ).

(b) Show that if we apply the unitary U to a system with density matrix ρ, then the density matrix of the transformed system is $U\rho U^\dagger$ (where $\dagger$ denotes conjugate transpose). Parts a and b, together, explain why the density matrix really does encode the experimentally accessible information about an ensemble—so if two ensembles have the same density matrix ρ, then no experiment can distinguish them.

(c) Show that if two ensembles have different density matrices, then there *is* an experiment that can distinguish them.

6. **No-communication theorem.** Quantum mechanics is a *local* theory, in the sense that not even entanglement can be used to communicate information faster than light. However, this is not quite immediate but is a theorem that has to be proved.

(a) Consider a state $|\psi\rangle = \sum_{i,j=1}^{n} \alpha_{ij}|i\rangle|j\rangle$, which involves two registers (possibly entangled with each other). Explain why no operation that we perform on the first register only (including unitaries and measurements), can affect the probability of any outcome of a standard-basis measurement on the second register only. [*Hint:* Write out the α_{ij}'s as an $n \times n$ matrix. What is the effect of an operation on the first register only?]

(b) Show that unitary operations on separate subsystems commute with each other: that is, $(U \otimes I)(I \otimes V) = (I \otimes V)(U \otimes I)$ for all U, V.

(c) Combining parts a. and b. conclude that no unitary transformation or measurement performed on the first register only, can affect the outcome of an experiment on the second register only.

7. **Conjugating CNOT.** Show that if you apply Hadamard gates to qubits A and B, followed by a CNOT gate from A to B, followed by Hadamard gates to A and B again, the end result is the same as if you had applied a CNOT gate from B to A. [*Note:* The CNOT gate won't be defined until the second lecture, so this problem is intended for after that.]

The above illustrates a principle of quantum mechanics you may have heard about: that any physical interaction by which A influences B can also cause B to influence A (so for example, it is impossible to measure a particle's state without affecting it).