Homework 6

Introduction to Quantum Information Science
Due Sunday, March 19th at 11:59 PM

1. In this problem, you'll explore something that we said in class about the Many-Worlds Interpretation of quantum mechanics: namely, that "two branches interfere with each other if and only if they produce an outcome that's identical in all respects." Consider the n-qubit "Schrodinger cat state" (or "generalized GHZ state")

$$\frac{|0\cdots 0\rangle + |1\cdots 1\rangle}{\sqrt{2}}$$

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a) What probability distribution over n-bit strings do we observe if we Hadamard the first $n-1$ qubits, then measure all n qubits in the $\{|0\rangle, |1\rangle\}$ basis?

b) Is this the same distribution or a different one, than if we had applied the same measurement to the state:

$$\frac{|0\cdots 0\rangle\langle 0\cdots 0| + |1\cdots 1\rangle\langle 1\cdots 1|}{2}$$

c) What probability distribution over n-bit strings do we observe if we Hadamard *all* n qubits, then measure all n qubits in the $\{|0\rangle, |1\rangle\}$ basis?

d) Is this the same distribution or a different one, than if we had applied the same measurement to the state:

$$\frac{|0\cdots 0\rangle\langle 0\cdots 0| + |1\cdots 1\rangle\langle 1\cdots 1|}{2}$$

2. Suppose Alice and Bob share a "noisy Bell pair": that is, a 2-qubit mixed state ρ, which consists of a Bell pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ with probability $1-\epsilon$, or an equal mixture of $|00\rangle$ and $|11\rangle$ with probability ϵ. Work out an expression for the probability with which Alice and Bob win the CHSH game, if they measure ρ using the usual strategy, the one that causes them to win with probability $\cos^2\left(\frac{\pi}{8}\right)$ using a perfect Bell pair. How large does ϵ need to be, before Alice and Bob's success probability using this strategy is no better than what it would be in the unentangled case? [Recall: In the CHSH game, Alice and Bob receive independent random bits x and y respectively. Their goal is to output bits $a = a(x)$ and $b = b(y)$ respectively such that $a + b = xy \pmod 2$. In the "usual strategy," Alice does nothing to her qubit if $x = 0$, and applies a $\frac{\pi}{4}$ counterclockwise rotation if $x = 1$. Bob applies a $\frac{\pi}{8}$ counterclockwise rotation if $y = 0$, or a $\frac{\pi}{8}$ clockwise rotation if $y = 1$. Then Alice and Bob both measure their qubits in the $\{|0\rangle, |1\rangle\}$ basis and output whatever they see.]

3. Which of the following 2-qubit density matrices represent separable states, and which ones represent entangled states? Prove your answer in each case. *Hint: Keep in mind that even a state that looks entangled might be separable because of a non-obvious decomposition as a mixture of product states. You're welcome to use Problem 2 as a way of proving that a state is entangled.*

a)

$$\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b)

$$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c)

$$\frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

d)

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

4. In the "GHZ game," there are three players, Alice, Bob, and Charlie, who are given bits x, y, and z respectively. We're promised that $x + y + z = 0 \pmod 2$; otherwise the bits could be arbitrary. The players' goal is, without communicating with each other, to output bits a, b, c respectively such that $a + b + c (\text{mod } 2) = \text{OR}(x, y, z)$. In other words, the parity of their output bits should be odd, if and only if at least one of the input bits is nonzero.

a) Show that, in a classical universe, there is no strategy that causes the players to succeed with certainty, for all four possible allowed inputs (x, y, z).

b) Now suppose that the players share the state:

$$\frac{|000\rangle - |011\rangle - |101\rangle - |110\rangle}{2}$$

Suppose that each player measures their qubit in the $\{|0\rangle, |1\rangle\}$ basis if their input bit is 0, or in the $\{|+\rangle, |-\rangle\}$ basis if their input bit is 1, and that they output according to what they see. Show that this lets the players win the GHZ game with certainty, for all four possible input triples.

Note that, in contrast to the two-player Bell/CHSH game, the GHZ game has the property that the quantum strategy wins with *certainty* — not just with greater probability than does any possible classical strategy. For that reason, the GHZ game is sometimes called a "magic" game or a "pseudo-telepathy" game.

c) Extra credit. Design a protocol that works for the GHZ state:

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}}$$