Homework 5

Introduction to Quantum Information Science
Due Friday, February 26 at 11:59 PM

1. We said in class that dense quantum coding requires 1 ebit of entanglement between Alice and Bob, in addition to 1 qubit of communication. In this problem, however, we'll see how to do a "poor man's" dense quantum coding with no entanglement, just 1 qubit of communication from Alice to Bob.

Suppose Alice knows two bits, x and y. She'd like to send a single qubit to Bob, which will let Bob learn either bit of his choice, x or y, though not necessarily both of them (and she doesn't know which Bob is interested in).

a) Describe a protocol that lets Bob learn the bit of his choice with $\cos^2(\pi/8) \approx 85\%$ success probability. *Hint: You might find the following states useful:*

$$\cos(\pi/8)\left|0\right\rangle + \sin(\pi/8)\left|1\right\rangle; \qquad \sin(\pi/8)\left|0\right\rangle + \cos(\pi/8)\left|1\right\rangle$$

$$\cos(\pi/8)\left|0\right\rangle - \sin(\pi/8)\left|1\right\rangle; \qquad \sin(\pi/8)\left|0\right\rangle - \cos(\pi/8)\left|1\right\rangle$$

b) Now suppose Alice is limited to classical communication only. And suppose also that the bits x and y are uniformly random and independent of each other. Describe a protocol that lets Bob learn the bit of his choice with 75% success probability.

2. A "qutrit" has the form $a\,|0\rangle + b\,|1\rangle + c\,|2\rangle$, where $|a|^2 + |b|^2 + |c|^2 = 1$. Suppose Alice and Bob share the entangled state $(|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$. Then consider the following protocol for teleporting a qutrit $|\psi\rangle = a\,|0\rangle + b\,|1\rangle + c\,|2\rangle$ from Alice to Bob: first Alice applies a CSUM gate from $|\psi\rangle$ onto her half of the entangled pair, where

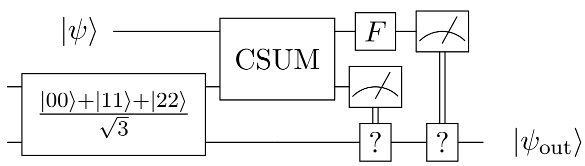
$$\mathrm{CSUM}(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |y + x \pmod 3\rangle\,.$$

Next, Alice applies the unitary matrix F to the $|\psi\rangle$ register, where

$$F = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix},$$

and $\omega = e^{2i\pi/3}$ so that $\omega^3 = 1$. She then measures both of her qutrits in the $\{|0\rangle, |1\rangle, |2\rangle\}$ basis, and sends the results to Bob over a classical channel. Show that Bob can recover a local copy of $|\psi\rangle$ given these measurement results.

This quantum circuit summarizes the protocol:



Here the double-lines represent classical 'trits' being sent from Alice to Bob. Depending on the value of 0,1 or 2 Bob can apply a ? gate 0, 1, or 2 times. Prove that $|\psi\rangle = |\psi_{\text{out}}\rangle$ for appropriately chosen ? gates for all possible measurement results. *Hint: You* could *explicitly work out all 9 possible cases, but you could also save time by noticing a general pattern that lets you handle all the cases in a unified way.*

3. Let:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle - \frac{1}{\sqrt{6}}|11\rangle$$

a) Put this state into Schmidt form:

$$\psi = \sum_i \lambda_i |\alpha_i\rangle |\beta_i\rangle$$

In other words, find orthonormal bases $\{|\alpha_0\rangle, |\alpha_1\rangle\}$ for the first qubit and $\{|\beta_0\rangle, |\beta_1\rangle\}$ for the second qubit, such that you can write the state without cross terms $|\alpha_0\rangle |\beta_1\rangle$ or $|\alpha_1\rangle |\beta_0\rangle$.

b) Calculate how many ebits of entanglement this state has. (Keep in mind, this answer need not be an integer.)

4. Suppose Alice and Bob hold one qubit each of an arbitrary two-qubit state $|\psi\rangle$ that is possibly entangled. They can apply local operations and are allowed classical communication. Their goal is to apply the CNOT gate to their state $|\psi\rangle$. How can they achieve this using two ebits of entanglement?