Homework 2

Introduction to Quantum Information Science
Due Sunday, February 5 at 11:59 PM

1. More fun with matrices.

a) Give an example of a 2x2 unitary matrix where the diagonal entries are 0 but the off-diagonal entries are nonzero.

b) Give an example for a 4x4 unitary matrix.

c) Is it possible to have a 3x3 unitary matrix with this condition? If no, prove it!

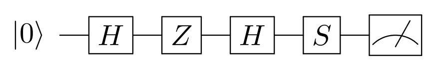
2. Single Qubit Quantum Circuits.

For the following circuits, calculate the output state before the measurement. Then calculate the measurement probabilities in the specified basis. Here we use:

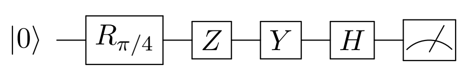
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad Y = iXZ$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad R_{\pi/4} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}; \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}; \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

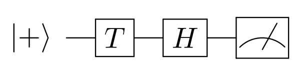
a) Measure in the $\{|0\rangle, |1\rangle\}$ basis:



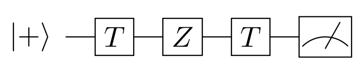
b) Measure in the $\{|+\rangle, |-\rangle\}$ basis:



c) Measure in the $\{|+\rangle, |-\rangle\}$ basis:



d) Measure in the $\{|i\rangle, |-i\rangle\}$ basis:



3. Miscellaneous.

a) Normalize the state $|0\rangle + |+\rangle$.

b) We say a quantum state vector $|\psi\rangle$ is an eigenvector or eigenstate of a matrix Λ if the following equation holds for some number λ:

$$\Lambda |\psi\rangle = \lambda |\psi\rangle$$

λ is called the eigenvalue of $|\psi\rangle$. Show that the state you normalized in part a) is an eigenstate of the H gate. What is the eigenvalue?

c) What single-qubit states are reachable from $|0\rangle$ using only H and S? Are there finitely or infinitely many?

4. Distinguishability of states.

Say you are given a state $|\psi\rangle$ that is either $|0\rangle$ or $|1\rangle$ but you don't know which. You can distinguish the two via a measurement in the $\{|0\rangle, |1\rangle\}$ basis.

a) But what if $|\psi\rangle$ is either $|0\rangle$ or $|+\rangle$ (with equal probability)? Give the protocol that distinguishes the two states with the optimal failure probability. Calculate this failure probability.

b) What is the failure probability if you measure in the $\{|0\rangle, |1\rangle\}$ basis?