Homework 11

Introduction to Quantum Information Science
Due Thursday, May 4th at 11:59 PM

1. Suppose we run the adiabatic algorithm with initial Hamiltonian H_i and final Hamiltonian H_f:

$$H_i = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad H_f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

At intermediate time $0 < t < 1$, the Hamiltonian that we apply is $H_t = (1-t)H_i + tH_f$.

a) Give the eigenstates of H_i and their associated energies.

b) Give the eigenstates of H_f and their associated energies.

c) Now give explicit expressions for the two eigenvalues of H_t, as a function of t. (You don't need to calculate the associated eigenvectors.) Plot these two eigenvalues on a graph, in the range $0 \le t \le 1$. (You can either use software, or just draw the graph by hand – as long as things look qualitatively right, we won't penalize you for imprecisions in drawing.)

d) At what time t is the minimum gap between the two eigenvalues achieved? What is the numerical value of that gap?

e) Based on this behavior, would you expect the adiabatic algorithm to reach the ground state of H_f, assuming it starts in the ground state of H_i, and the linear interpolation between H_i and H_f is sufficiently slow?

f) Now suppose instead that we set:

$$H_f = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

with everything else exactly as before. Now plot the two eigenvalues of H_t on a graph, in the range $0 \le t \le 1$.

g) What is the minimum eigenvalue gap in this case, and where is it achieved? In this case, do you expect the adiabatic algorithm to reach the ground state of H_f?

2. In class, we saw the simplest classical error-correcting code, which encodes one bit into three bits via $\bar{0} = 000$ and $\bar{1} = 111$, and which corrects any single bit-flip error. We then saw the quantum generalization of that code, the Shor 9-bit code, which encodes 1 qubit into 9 qubits via:

$$|\bar{0}\rangle = \frac{(|000\rangle + |111\rangle)^{\otimes 3}}{2\sqrt{2}}; \quad |\bar{1}\rangle = \frac{(|000\rangle - |111\rangle)^{\otimes 3}}{2\sqrt{2}}$$

and which corrects any single bit-flip or phase-flip error – and therefore, by linearity, any single-qubit error at all.

a) Prove that any classical encoding of a single bit, which corrects an arbitrary single bit-flip error, requires at least 3 bits for the codewords. I.e., there is no such code using 2 bits.

b) A weaker notion than an error-correcting code is an error-*detecting* code – one that detects whether an error has occurred, though not necessarily how to correct the error. Give a classical error-detecting code, which encodes 1 bit into 2 bits, and which detects a single bit-flip error.

c) Now give a quantum error-detecting code, which encodes 1 qubit into 2 qubits, and which detects a single phase-flip error.

d) Give a quantum error-detecting code, which encodes 1 qubit into 4 qubits, and which detects a single bit-flip error *or* a single phase-flip error. [Hint: Adapt the Shor code.]

e) Draw the quantum circuits for detecting bit-flip and phase-flip errors using your code from part d), and explain why they work.

f) Consider the quantum code $|\bar{0}\rangle = |00\rangle$ and $|\bar{1}\rangle = |11\rangle$. Give an example of a qubit encoded using this code, as well as a single-qubit error on the encoded qubit, such that this code fails to detect even that an error has occurred.

3. In class, we saw the stabilizer formalism, whereby a certain class of n-qubit quantum states important for error-correction – namely, the *stabilizer states* – can be specified using only $n(2n+1)$ classical bits each. Here a stabilizer state is any state reachable from the all-0 state by applying a circuit composed of CNOT, Hadamard, and Phase gates only.

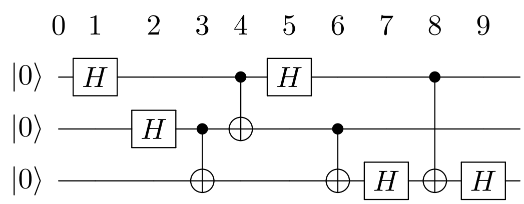
We also gave a set of rules for updating the classical description of a stabilizer state whenever a CNOT, Hadamard, or Phase gate is applied to it. Briefly, and ignoring the effects on the +/- phase bits, the rules were:

CNOT from qubit i to qubit j: Add column i into column j in the X matrix, and add column j into column i in the Z matrix

Hadamard on qubit i: Swap column i in the X matrix with column i in the Z matrix

Phase on qubit i: Add column i in the X matrix into column i in the Z matrix

Now consider the following quantum circuit, which acts on 3 qubits that are initially in the state $|000\rangle$.



This circuit abstractly represents quantum teleportation from the first qubit to the third qubit.

a) Let $|\psi_t\rangle$ be the state of the 3 qubits immediately after the t-th gate is applied in this circuit. Write out each of the states $|\psi_0\rangle, ..., |\psi_9\rangle$, using ordinary ket notation (you're allowed to use the notations $|+\rangle$ and $|-\rangle$).

b) Now write out each of $|\psi_0\rangle, ..., |\psi_9\rangle$ using stabilizer tableaus, consisting of two 3×3 binary matrices side by side. [Hint: With this particular circuit, the +/- sign bits will always remain set to +.]

To get you started: recall that $|\psi_0\rangle = |000\rangle$ can be represented by the products of Pauli matrices $\{+ZII, +IZI, +IIZ\}$ which corresponds to the stabilizer tableau:

$$\left[\begin{array}{c|c|c} + & 000 & 100 \\ + & 000 & 010 \\ + & 000 & 001 \end{array}\right]$$

Then $|\psi_1\rangle = |+\rangle|0\rangle|0\rangle$ can be represented by $\{+XII, +IZI, +IIZ\}$ which corresponds to the stabilizer tableau

$$\left[\begin{array}{c|c|c} + & 100 & 000 \\ + & 000 & 010 \\ + & 000 & 001 \end{array}\right]$$

In general, we strongly recommend that you apply the rules for CNOT and Hadamard directly to the stabilizer tableaus, rather than trying to convert the states from part a) into stabilizer notation. However, do check that the final results that you got, in parts a) and b), represent the same quantum state!