Homework 1
Introduction to Quantum Information Science
Due Sunday, January 29th at 11:59 PM

1. Stochastic and Unitary Matrices.
   a) Of the following matrices, which ones are stochastic? Which ones are unitary?

   \[
   A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},
   \]
   \[
   E = \begin{pmatrix} 2 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}, \quad F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{3i}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3i}{5} \end{pmatrix}.
   \]

   b) Show that any stochastic matrix that’s also unitary must be a permutation matrix.

   c) Stochastic matrices preserve the 1-norms of nonnegative vectors, while unitary matrices
   preserve 2-norms. Give an example of a 2 × 2 matrix, other than the identity matrix, that
   preserves 4-norm of real vectors \((a b)^T\): that is, \(a^4 + b^4\).

   d) [Extra credit] Give a characterization of all matrices that preserve the 4-norms of real
   vectors. Hopefully, your characterization will help explain why preserving the 2-norm, as
   quantum mechanics does, leads to a much richer set of transformations than preserving
   the 4-norm does.

2. Tensor Products.
   a) Calculate the tensor product \(\left( \begin{array}{c} 2 \\ 3 \end{array} \right) \otimes \left( \begin{array}{c} \frac{1}{3} \\ 2 \end{array} \right)\).

   b) Of the following length-4 vectors, decide which ones are factorizable as 2 × 2 tensor
   products, and factorize them. (Here the vector entries should be thought of as labeled by
   00, 01, 10, and 11 respectively.)

   \[
   A = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{2} \\ \frac{1}{4} \\ \frac{9}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.
   \]

3. Prove that there’s no 2 × 2 real matrix \(A\) such that

   \[A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
   \]
This observation perhaps helps to explain why the complex numbers play such a central role in quantum mechanics.

4. **Dirac notation.**

a) Let $|\psi\rangle = \frac{|0\rangle + 2|1\rangle}{\sqrt{5}}$ and $|\phi\rangle = \frac{2i|0\rangle + 3|1\rangle}{\sqrt{13}}$. What’s $\langle\psi|\phi\rangle$?

b) Usually quantum states are normalized: $\langle\psi|\psi\rangle = 1$. The state $|\phi\rangle = 2i|0\rangle - 3i|1\rangle$ is not normalized. What constant $A$ makes $|\psi\rangle = \frac{|\phi\rangle}{A}$ a normalized state?

c) Define $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ and $|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$. Show explicitly that the vectors $|i\rangle$ and $|-i\rangle$ form an orthonormal basis for $\mathbb{C}^2$.

d) Write the normalized vector $|\psi\rangle$ from part b in the $\{|i\rangle, |-i\rangle\}$ basis.