Homework 1

Introduction to Quantum Information Science
Due Sunday, January 29th at 11:59 PM

1. Stochastic and Unitary Matrices.

a) Of the following matrices, which ones are stochastic? Which ones are unitary?

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$
$$E = \begin{pmatrix} 2 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}, \quad F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{3i}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3i}{5} \end{pmatrix}$$

b) Show that any stochastic matrix that's also unitary must be a permutation matrix.

c) Stochastic matrices preserve the 1-norms of nonnegative vectors, while unitary matrices preserve 2-norms. Give an example of a 2×2 matrix, other than the identity matrix, that the preserves 4-norm of real vectors $\begin{pmatrix} a \\ b \end{pmatrix}$: that is, $a^4 + b^4$.

d) [Extra credit] Give a characterization of all matrices that preserve the 4-*norms* of real vectors. Hopefully, your characterization will help explain why preserving the 2-norm, as quantum mechanics does, leads to a much richer set of transformations than preserving the 4-norm does.

2. Tensor Products.

a) Calculate the tensor product $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$.

b) Of the following length-4 vectors, decide which ones are factorizable as 2×2 tensor products, and factorize them. (Here the vector entries should be thought of as labeled by 00, 01, 10, and 11 respectively.)

$$A = \begin{pmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{4}{9} \\ \frac{2}{9} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

3. Prove that there's no 2×2 *real* matrix A such that

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This observation perhaps helps to explain why the complex numbers play such a central role in quantum mechanics.

4. Dirac notation.

a) Let $|\psi\rangle = \frac{|0\rangle + 2|1\rangle}{\sqrt{5}}$ and $|\phi\rangle = \frac{2i|0\rangle + 3|1\rangle}{\sqrt{13}}$. What's $\langle\psi|\phi\rangle$?

b) Usually quantum states are normalized: $\langle\psi|\psi\rangle = 1$. The state $|\phi\rangle = 2i\,|0\rangle - 3i\,|1\rangle$ is not normalized. What constant A makes $|\psi\rangle = \frac{|\phi\rangle}{A}$ a normalized state?

c) Define $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ and $|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$. Show explicitly that the vectors $|i\rangle$ and $|-i\rangle$ form an orthonormal basis for $\mathbb{C}^2$.

d) Write the normalized vector $|\psi\rangle$ from part b in the $\{|i\rangle, |-i\rangle\}$ basis.