Lecture 3, Tues Jan 24: Basic Rules of QM

Tensor products are a way of building bigger vectors out of smaller ones.

Let's apply a NOT operation to the first bit, and do nothing to the second bit. That's really the same as defining the function f as $f(00) = 10, f(01) = 11, f(10) = 00, f(11) = 01$. So we can write the operation as follows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{array}{c} {}^{00}(\\ {}^{01}(\\ {}^{10}(\\ {}^{11}(\\ \end{array} \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ {\scriptstyle 00} & {\scriptstyle 01} & {\scriptstyle 10} & {\scriptstyle 11} \end{array} \begin{array}{c})^{00} \\ )^{01} \\ )^{10} \\ )^{11} \\ \end{array}$$

A **quantum state** (technically, a "pure state") is a unit vector in $\mathbb{C}^N$ describing the state of a quantum system.

Formally a quantum state could exist in any dimension. Physics courses cover infinite-dimensional vectors, but we'll stick to discrete systems (which is to say that when we make a measurement, there are only finitely many possible outcomes).

What does quantum mechanics say about the universe being discrete or continuous at the base level? It suggests a strange, hybrid picture. There's a continuum of possible quantum states, but every measurement has a discrete outcome. A system with two amplitudes, $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, has uncountably infinitely many possible states (given the only restriction is that $|\alpha|^2 + |\beta|^2 = 1$), though note that the same would be true even if we described states using classical probabilities. In both cases, classical and quantum, the continuum is never directly observed, but is only used to calculate the probabilities of discrete outcomes.

The **qubit** is the simplest interesting quantum system.
It's a two-level system (we label the levels '0' and '1'), with an amplitude for 0 and an amplitude for 1.

A one-state quantum system would just be $\begin{bmatrix} 1 \end{bmatrix}$. Not very interesting!

In physics, following Dirac, we like to write quantum state vectors using the so-called **Ket Notation**.

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}^{0}_{1} = \alpha|0\rangle + \beta|1\rangle$$

Note that $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and that $|\psi\rangle$ is a symbol we'll often use for quantum states.

Why do we use ket notation?

One main advantage is that practically speaking, we usually deal with really sparse vectors (where most amplitudes are 0). Ket notation makes it easier to represent only the values we're talking about.

It's really just a formalism to make life easier, we can put anything in ket notation. Look: this is Schrödinger's Cat in ket notation: $|$  $\rangle + |$  $\rangle$.

Often you'll need to take the transpose of a vector $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \rightarrow \begin{bmatrix}\alpha & \beta\end{bmatrix}$ or for complex values $\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \rightarrow \begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}$

Using the complex conjugate allows you to define a norm
$||v||^2 = v^T v$

Then we get $v^T v = \begin{bmatrix}\alpha^* & \beta^*\end{bmatrix} \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

$$= \alpha^*\alpha + \beta^*\beta$$
$$= |\alpha|^2 + |\beta|^2$$

What does this look like in ket notation?

Just like we have the **ket** $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ for $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

We define the **bra** $\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$ for $\begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}$

And we define $\langle x|y\rangle$ as the inner product of $|x\rangle$ with $|y\rangle$ (which automatically involves taking the conjugate transpose of $|x\rangle$).

Therefore $\langle\psi|\psi\rangle = 1$.
So $\langle v|w\rangle = \langle w|v\rangle^*$.

Remember: the way we change quantum states is by applying linear transformations:

$$U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}$$

A linear transformation is **unitary** if $|\alpha|^2 + |\beta|^2 = |\alpha'|^2 + |\beta'|^2$.

Unitary matrices correspond to unitary transformations.

We've got the identity $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and permutation $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ matrices, which are the only unitaries that are also stochastic.

Other unitaries include

$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$ which maps $|0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow i|1\rangle$, $\begin{bmatrix}0 & -i\\ -1 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 1 & e^{i\theta}\end{bmatrix}$

Note: Euler's Equation says $e^{i\theta} = \cos\theta + i\sin\theta$

All real possible states of a qubit define a circle and all complex possible states define a hypersphere. That's because these states are all the vectors of length 1.

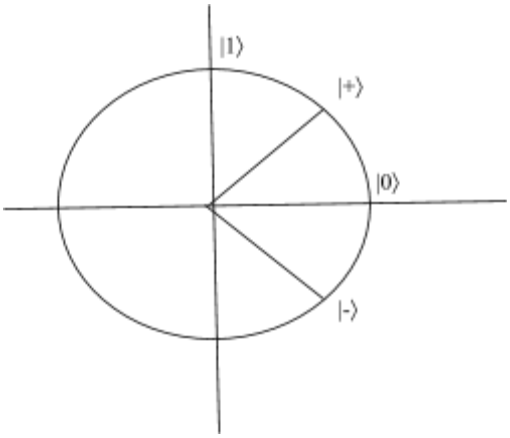
We define:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$



Unitary transformations are norm-preserving linear transformations.

For any angle θ you could have $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ which grabs a vector and rotates it θ radians.

For example $R_{\pi/4} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

To get the transform of $(ABv)^\dagger$ $v^\dagger B^\dagger A^\dagger$

What does it mean that a unitary matrix preserves the 2-norm?

It means applying a unitary transformation retains the inner product, $\langle\psi|\psi\rangle$.

$$\langle\psi|\psi\rangle = (U|\psi\rangle)^\dagger U|\psi\rangle = \langle\psi|U^\dagger U|\psi\rangle$$

For this to hold for any ψ, $U^\dagger U$ must equal I. Which means $U^{-1} = U^\dagger$.

That in turn implies that the rows of U must be an orthogonal unit basis.

So you can tell if a matrix is unitary by checking if the rows (or, equivalently, the columns!) form an orthogonal unit basis.

This is not the "operational definition" of unitary matrices, but is a logical consequence of unitary transformations preserving inner products.

An **orthogonal matrix** is both unitary and real.

Any orthogonal matrix is a product of rotations and reflections.

Some examples:

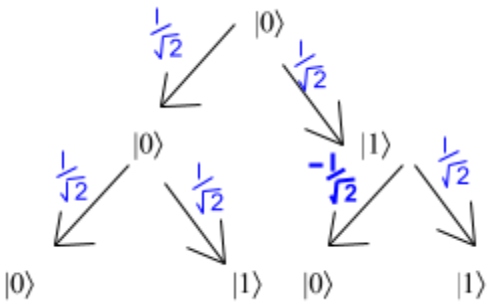
$R_{\pi/4}|0\rangle = |+\rangle$

$R_{\pi/4}|+\rangle = |1\rangle$
$R_{\pi/4}|1\rangle = -|-\rangle$

You'll get a full revolution after applying $R_{\pi/4}$ eight times.

In the classical world, if an event could happen multiple ways, but will be "random" no matter which way it happens, then it's simply "random" overall.
But in the quantum world, you can sometimes apply a unitary transformation to a superposition state and then get a determinate answer

Anything interesting in quantum mechanics can be explained in terms of **interference**.



The $|0\rangle$ basis state can go to states $|0\rangle$ and $|1\rangle$ equally.

Above, there are two different paths that lead to the $|0\rangle$ outcome, but they cancel each other out, with one having positive amplitude and the other having negative amplitude.

The $|0\rangle$ states interfere destructively.
The $|1\rangle$ states interfere constructively.

No matter what unitary transformation you apply: If $|0\rangle$ goes to $U|0\rangle$, then $-|0\rangle$ goes to $-U|0\rangle$.

The zero state and the minus zero state are indistinguishable mathematically, which is to say:

Global phase is unobservable.

Multiplying your entire quantum state by a scalar is like if last night someone moved the entire universe twenty feet to the left. We can only really measure things relative to other things! Which leads to a second maxim:

Relative phase *is* observable.

To distinguish between the states $|+\rangle$ and $|-\rangle$ we can rotate by 45 degrees and then measure to see whether we got $|0\rangle$ or $|1\rangle$.

There are no second chances. Once you measure, the outcome is set.