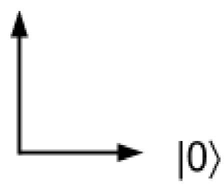
Lecture 14, Thurs March 2: Nonlocal Games

Last time we talked about the CHSH Game, and how no classical strategy lets Alice and Bob win it more than 75% of the time. Today we'll see how, by using entanglement, they can win 85% of the time—and then we'll delve deeper to try to understand what's going on.

The strategy involves Alice and Bob measuring their respective qubits in different bases, depending on whether their input bits *x* and *y* are 0 or 1, and then outputting bits *a* and *b* respectively based on the outcomes of those measurements.

Let $|\frac{\pi}{8}\rangle = \cos(\frac{\pi}{8})|0\rangle + \sin(\frac{\pi}{8})|1\rangle$, and let $|-\frac{\pi}{8}\rangle = \cos(-\frac{\pi}{8})|0\rangle + \sin(-\frac{\pi}{8})|1\rangle$.

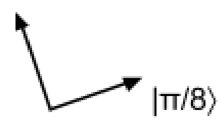
If $x = 0$, Alice measures in



and if $x = 1$, Alice measures in

She sets a to 0 if she measures $|0\rangle$ or $|+\rangle$
and 1 if she measures $|1\rangle$ or $|-\rangle$

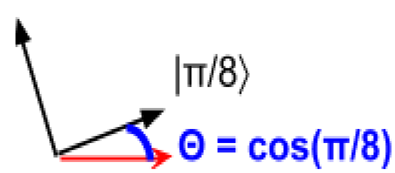
If $y = 0$, Bob measures in the X basis rotated by $\frac{\pi}{8}$ clockwise.



and if $y = 0$ rotated by $-\frac{\pi}{8}$,

he sets b to 0 if he measures $|\frac{\pi}{8}\rangle$ or $|-\frac{\pi}{8}\rangle$
1 if otherwise

This strategy has the amazing property of making Alice and Bob win with probability $\cos^2(\frac{\pi}{8})$ for all possible values of *x* and *y*.



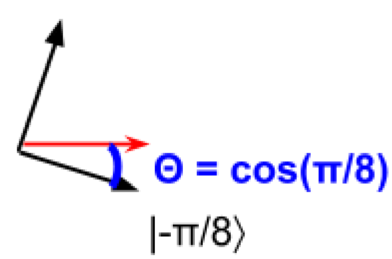
So why does this strategy work 85% of the time?

Let's consider the case where Alice gets $x = 0$ and measures $|0\rangle$.
She'll output $a = 0$, and she and Bob will win iff Bob outputs $b = 0$.

So what are the odds that Bob outputs 0?

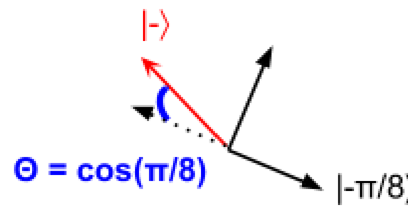
Given that Alice measured her qubit already, Bob's qubit collapsed to the $|0\rangle$ state.

First suppose $y = 0$. Then Bob measures the $|0\rangle$ state in a basis rotated by $\frac{\pi}{8}$ clockwise. He outputs 0 if he measures $|\frac{\pi}{8}\rangle$. Thus, the probability that Bob outputs 0 in this case is $\cos^2(\frac{\pi}{8}) \approx 85\%$.



We can do the same calculation for the case $y = 1$. The angle between vectors is still $\frac{\pi}{8}$. In fact, we can generalize this result to *all* the cases where either x or y is 0.

Note that we can assume without loss of generality that Alice measures first, because of the No Communication Theorem.



The interesting case is where a and b are both 1.

Here Alice measures in the $\{|+\rangle, |-\rangle\}$ basis. Assume without loss of generality that Alice gets the outcome $|-\rangle$. Then what's Bob's probability of getting the outcome $|-\frac{\pi}{8}\rangle$?

It's still $\cos^2(\frac{\pi}{8})$, because the angle between $|-\rangle$ and $|-\frac{\pi}{8}\rangle$ is $\frac{\pi}{8}$, and global phase doesn't matter.

So, Alice and Bob win the game with probability $\cos^2(\frac{\pi}{8})$ in all four cases.

How does this game relate to hidden variable theories? Well, if all correlations between the qubits could be explained by stories like "if anyone asks, we're both 0," then we'd make a firm prediction: that Alice and Bob can win the CHSH game at most 75% of the time (because that's how well they can do by pre-sharing arbitrary amounts of classical information).

So if they play the game repeatedly, and demonstrate that they can win more than 75% of the time, then local realism is false. Notice that nowhere in this argument did we ever need to presuppose that quantum mechanics is true.

<u>Does Alice and Bob's ability to succeed more than 75% of the time mean that they are communicating?</u>

Well, we know it's not possible for either to send a signal to the other, by the No-Communication Theorem. But how can we reconcile that with their success in the CHSH game?

One way to understand what's going on, is to work out Alice and Bob's density matrices explicitly.

Bob's initial density matrix is $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and after Alice measures it's still $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

So in that sense, no signal has been communicated from Alice to Bob. Nevertheless, *if* you knew both Alice's measurement and its outcome, then you could update Bob's density matrix to that of a pure state. That shouldn't worry us though, since even classically, if you condition on what Alice sees then you can change your predictions for Bob.

Imagine a hierarchy of possibilities for what the universe allows. <u>Classical Local Realism</u> is at the bottom: that's where you only ever need to use classical probability theory when you have incomplete information about physical systems, *and also* signals propagate only at the speed of light.

At the top of the hierarchy is the <u>Faster-Than-Light Science-Fiction Utopia</u>, where Alice and Bob can communicate instantaneously, you can travel faster than light, and so forth.

A priori, people tend to believe that reality must be one or the other. So when they read pop-science articles about how classical local realism is false, they think, "OK, then we must live in the FTL sci-fi utopia."

Instead, the truth—according to quantum mechanics—is in the middle, and is so subtle that perhaps no science-fiction writer would ever have had the imagination to invent it. We live in a world where there's no classical local realism, but no faster-than-light communication either. Or to put it another way: a *classical simulation* of our universe would involve FTL communication, but our universe itself does not.

Maybe no science fiction writer ever came up with this possibility, simply because it's hard to think of a plot that requires Alice and Bob to win the CHSH game 85% of the time instead of 75%!

Indeed, this is a key piece of evidence that our world really *is* quantum, and is not secretly classical behind the scenes: because we know from Bell that the latter possibility would require FTL communication.

So where is that $\cos^2(\frac{\pi}{8})$ *coming from anyways? It seems so arbitrary…*

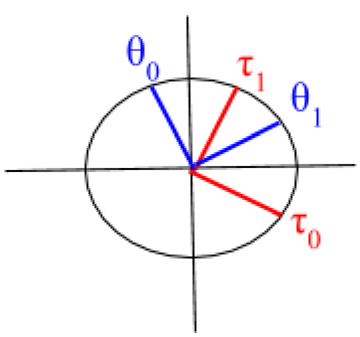
It may seem like the $\cos^2(\frac{\pi}{8})$ is simply coming from our particular approach to the problem. Maybe if we came at it another way, we could use entanglement to win *even more* than 85% of the time: why not 100%?

Surprisingly, $\cos^2(\frac{\pi}{8})$ turns out to be optimal, even if Alice and Bob share unlimited amounts of entanglement. This is the upshot of

Tsirelson's Inequality

...a cousin of the Bell inequality, proved in the 1980s.

It requires a *bit* too much machinery to give a complete proof of Tsirelson's Inequality here. However, we'll convey the intuition, by showing that, among strategies "similar to the one we used," ours was the optimal one.



Let's say that Alice has two angles:

θ_0, the angle she measures in if she receives input $x = 0$, and

θ_1, the angle she measures in if she receives input $x = 1$.

Similarly, Bob has τ_0 and τ_1, corresponding to $y = 0$ and $y = 1$ respectively .

The same rules apply from the solution we constructed earlier for the CHSH game. All we're doing here is changing the chosen vectors into variables to try and show that there's no better vectors to chose than the ones we did.

The key formula is this: Alice and Bob's total success probability is

$$P[\text{success}] = \frac{1}{4}[\cos^2(\theta_0 - \tau_0) + \cos^2(\theta_0 - \tau_1) + \cos^2(\theta_1 - \tau_0) + \sin^2(\theta_1 - \tau_1)]$$

Why?

1. Each of the four input pairs has an equal chance of occurring.
2. In the first three cases, Alice and Bob win iff they output the same bit, so we take the squared cosine between their measurement angles.
3. But in the fourth case, $x = y = 1$, Alice and Bob win iff they output *different* bits. So in this case, we take the squared sine between their measurement angles.

Now we use some high-school trigonometry to get the above equals

$$\frac{1}{2} + \frac{1}{8}[\cos(2(\theta_0 - \tau_0)) + \cos(2(\theta_0 - \tau_1)) + \cos(2(\theta_1 - \tau_0)) - \cos(2(\theta_1 - \tau_1))]$$

We can get rid of the 2's inside the cosines, by simply realizing that we could adjust our original angles to account for them.

It will be helpful to think of the cosines as the inner products between unit vectors. In that case, we can rewrite the above as

$$\frac{1}{2} + \frac{1}{8}[u_0 \cdot v_0 + u_0 \cdot v_1 + u_1 \cdot v_0 - u_1 \cdot v_1]$$

$$= \frac{1}{2} + \frac{1}{8}[u_0 \cdot (v_0 + v_1) + u_1 \cdot (v_0 - v_1)]$$

Since u_0, u_1, v_0, and v_1 are all unit vectors, the above is upper-bounded by

$$\leq \frac{1}{2} + \frac{1}{8}\left[\|v_0 + v_1\| + \|v_0 - v_1\|\right]$$

From here, we can use the parallelogram inequality to bound it further:

$$\leq \frac{1}{2} + \frac{1}{8}\sqrt{2([\|v_0 + v_1\|^2 + \|v_0 - v_1\|^2)}$$

Which equals

$$\frac{1}{2} + \left(\frac{\sqrt{2}}{8}\right)\sqrt{4} = \frac{1}{4}\left(2 + \sqrt{2}\right)$$

Which wouldn't you know it, is equal to

$$\mathbf{cos^2}(\tfrac{\pi}{8}) \qquad \approx 85\%$$

So that really is the maximum winning probability for the CHSH game.

There's been a trend in the last 15 years to study theories that would go past quantum mechanics by letting you violate Tsirelson's Inequality, but that would still prohibit faster-than-light travel. In such a world, it's been proven (among other things) that if Alice and Bob wanted to schedule something on a calendar, they could decide if there's a date where they're both free by exchanging only one bit of communication. That's a lot better than can be done under the rules of quantum mechanics!

Testing the Bell Inequality

When Bell proved his inequality, he was just trying to make a conceptual point, about the necessity for nonlocal influences in any hidden-variable theory underlying quantum mechanics. But by the 1980s, technology had advanced to the point where playing the CHSH game was actually a feasible physics experiment! Alain Aspect (and others) ran the experiment, and the results were fully consistent with quantum mechanics, and extremely problematic for local hidden variable theories.

The experiments don't quite get to 85% success probability, given the usual difficulties that afflict quantum experiments. But you can reach a high statistical confidence that you're winning more than, say, 80% of the time.

This was evidence, not only that local realism was false, but also that <u>entanglement had been created</u>.

Most physicists shrugged, already sold on quantum mechanics (and on the existence of entanglement). But a few, still committed to a classical view of the world, continued to look for loopholes in the experiment.

Skeptics pointed out two loopholes in the existing experiments, essentially saying "if you squint enough, classical local realism might still be possible":

1. <u>Detector Inefficiency</u>

Sometimes detectors fail to detect a photon, or they detect non-existent photons. Enough noise in the experiments could skew the data.

2. <u>The Locality Loophole</u>

Performing the measurements and storing their results in a computer memory takes some time: maybe nanoseconds or microseconds. Now, unless Alice and Bob and the referee are *very* far away from each other, this opens the possibility of a sort of "local hidden variable conspiracy," where as soon as Alice measures, some particle (unknown to present-day physics) flies over to Bob and says "hey, Alice got the measurement outcome 0; you should return the measurement outcome 0 too." The particle would travel "only" at the speed of light, yet could still reach Bob before his computer registered the measurement outcome.

By the 2000s, physicists were able to close loophole 2, but only in experiments still subject to loophole 1. And conversely, they could close loophole 1, but only in experiments still subject to 2. Finally, in 2016, several teams managed to do experiments that closed both loopholes simultaneously.

There are still people who deny the reality of quantum entanglement, but through increasingly solipsistic arguments. For example…

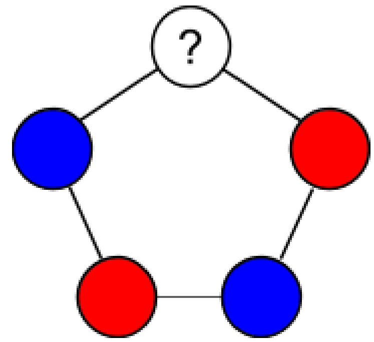
Superdeterminism

is a strange way to maintain that classical local realism is still the law of the land. Superdeterminism explains the results of CHSH experiments by saying "We only *think* Alice and Bob can choose measurement bases randomly. Actually, there's a grands cosmic conspiracy involving all of our brains, our computers, and our random number generators, with the purpose of rigging the measurement bases to ensure that Alice and Bob can win the CHSH game ~85% of the time. But that's all this cosmic conspiracy does: it doesn't allow FTL communication or anything like that, even though it easily could."

Nobel Laureate Gerard ‘t Hooft advocates superdeterminism, so it’s not like the idea lacks distinguished supporters, but Professor Aaronson is on board with entanglement.

Now we’ll look at some other non-local games, to see what else entanglement can help with. First we have…

The Odd Cycle Game



There’s a cycle with an odd number of vertices *n*.

Alice and Bob claim that they have a two-coloring of the cycle, but basic graph theory tells us that this isn’t possible.

Nevertheless, Alice and Bob will try to convince a referee that they’ve found a two-coloring anyway. They’ll do that by using entanglement to coordinate their responses to challenges from a referee.

The referee performs two obvious consistency checks:

- He can ask them both the color of a random vertex *v*, in the two-coloring they found.
  - They pass the test iff their answers are the same
- He can ask Alice the color of a random vertex *v*, and Bob the color of an adjacent vertex *w*.
  - They pass the test iff their answers are different

The referee chooses between these tests with equal probability—and crucially, he doesn’t tell Alice or Bob which test he’s performing.

In a single run of the game, the referee performs one such test, and gets answers from Alice and Bob. Without loss of generality, assume their answers are always RED or BLUE.

What strategy provides the best probability that Alice and Bob will pass the referee’s test and win the game?

Classically, we know that regardless of what Alice and Bob do, $Pr[\text{win}] < 1$.

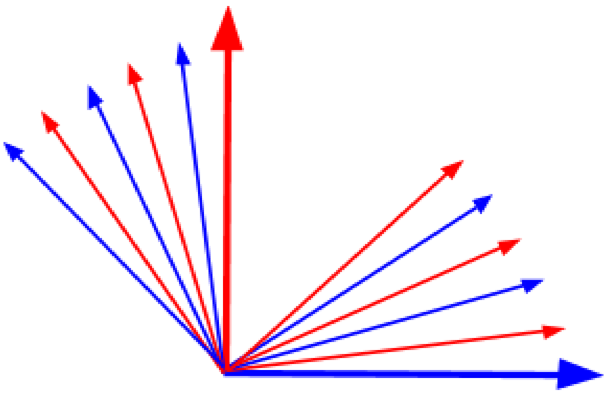
Why? Because for Alice and Bob to answer all possible challenges correctly, they’d need an actual two-coloring, which is impossible. The best they can do is agree on a coloring for all but one of the vertices, which gets them $Pr[\text{win}] = 1 - \frac{1}{2n}$.

Nevertheless, we claim that quantumly Alice and Bob can do better, and can achieve

$$Pr[\text{win}] \approx 1 - \frac{1}{n^2}.$$

if they pre-share a single Bell pair, $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

The strategy is as follows: Alice and Bob both measure their qubits in a basis depending on the vertex they're asked about.



The measurement bases for adjacent vertices differ by $\frac{2\pi}{n}$, so the bases rotate all the way around the unit circle.

The first basis has outcome $|0\rangle$ map to answering BLUE, and outcome $|1\rangle$ map to answering RED. The second basis has outcome $|\frac{2\pi}{n}\rangle$ map to answering RED, and outcome $|\frac{\pi}{2} + \frac{2\pi}{n}\rangle$ map to answering BLUE. They continue alternating in this way.

The upshot is that, when Alice and Bob are asked about the same vertex, they both measure in the same basis, and thus answer with the same color.

On the other hand, when Alice and Bob are asked about adjacent vertices, we get a similar situation to the CHSH game, where the probability of Bob producing the same output as Alice, is the squared sine of the angle between their vectors:

$$\sin^2 \theta = \sin^2 \left(\frac{1}{n}\right) \approx \frac{1}{n^2}.$$

Another crazy game is…

The Magic Square Game

		0
0	1	1
		0

Here Alice and Bob claim that they can fill a 3×3 grid with 0's and 1's so that:

1. Every row has an even sum
2. Every column has an odd sum

The referee asks Alice to provide a randomly-chosen row of the grid, and asks Bob to provide a randomly-chosen column. Alice and Bob "win" the game if and only if:

- The row has an even sum
- The column has an odd sum
- The row and column agree on the square where they intersect

You can see that constraints 1 and 2 *can't* actually both be satisfied, by examining the total number of 1's in the grid. Constraint 1 requires this total number to be even, while constraint 2 requires it to be odd. This implies that there's no classical strategy that lets Alice and Bob win the game with probability 1.

Nevertheless, David Mermin (the author of our textbook) discovered a quantum strategy where Alice and Bob win with probability 1. This strategy requires them to share 2 ebits of entanglement.

We won't describe the strategy in class, since it's very complicated to state without the "stabilizer formalism," which we haven't yet seen (but will by the end of the course).