Lecture 13, Tues Feb 28: Hidden Variables, Bell's Inequality

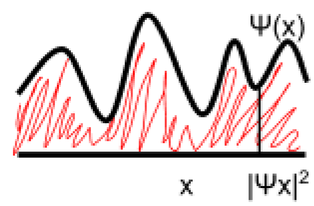
In the last lecture, we discussed four different attitudes people take toward quantum mechanics: Copenhagen, "shut up and calculate," dynamical collapse, and Everett's Many-Worlds Interpretation. You might think that *all* the options we've seen so far are bizarre and incomprehensible (Einstein certainly did), and wonder if we could come up with a theory that avoids all of the craziness. This leads us to the old dream of…

Hidden Variable Theories

which try to supplement quantum state vectors with some sort of hidden ingredients. The idea is to have a state, like $\alpha|0\rangle + \beta|1\rangle$, represent "merely" a way of making a prediction about what the universe *has already* set the result of measuring the qubit to be: either $|0\rangle$ or $|1\rangle$.

The most famous hidden-variable theory is...

Bohmian Mechanics



which was developed by David Bohm in the 1950s. It's also called the "deBroglie-Bohm theory," because it turns out that Louis de Broglie had the exact same idea in the 1920s—although deBroglie quickly disavowed it, after the idea faced a poor reception from other quantum mechanics pioneers.

Normal quantum mechanics says that a particle is in a superposition of locations, which we can use to calculate the probability that the particle will be found in one place or another when measured—and moreover, that this superposition exhausts what can be said about the particle's location. But, while keeping that superposition as part of physics, we now want to say that there's *also* a "real place" where the particle is, even before anyone measures it. To make that work, we need to give a rule for how the superposition "guides" the real particle. And this rule should have the property that, if anyone *does* measure the particle, they'll find exactly the result that quantum mechanics predicted for it---since we certainly don't want to give up on quantum mechanics' empirical success!

At first, you might think it would be tricky to find such a rule; indeed, you might wonder whether such a rule is possible at all. However, the real problem turns out to be more like an embarrassment of riches! There are infinitely many possible rules that could satisfy the above property—but by design, they all yield exactly the same predictions as standard quantum mechanics. So there's no experimental way to know which one is correct.

To explain this in a bit more detail, let's switch from particle positions back to the discrete quantum mechanics that we're more comfortable with in this course. Suppose we have a quantum pure state, represented as an amplitude vector in some fixed basis. Then when we multiply by a unitary transformation, suppose we want to be able to say: "this is the basis state we were *really in* before the unitary was applied, and this is the one we're *really in* afterwards." In other words, we want to take the equation

$$\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} U_{11} & & U_{1n} \\ & \ddots & \\ U_{n1} & & U_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

and map it to an equation

$$\begin{bmatrix} |\beta_1|^2 \\ \vdots \\ |\beta_n|^2 \end{bmatrix} = \begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} |\alpha_1|^2 \\ \vdots \\ |\alpha_n|^2 \end{bmatrix}$$

for some choice of stochastic matrix S (possibly depending on the input and output vectors).

There are many, many such matrices S. For example you could put $\begin{bmatrix} |\beta_1|^2 \\ \vdots \\ |\beta_n|^2 \end{bmatrix}$ in every column, which would say that you're always jumping randomly over time, but in a way that preserves the Born Rule. You could have been in a different galaxy a Planck time ago; now you're here (with fictitious memories planted in your brain); who knows where you'll be a Planck time from now?

But Bohm thought, not about this discrete setting, but mostly about the example of a particle moving around in continuous Euclidean space. And in the latter case, it turns out that one can do something nice that isn't possible with finite-dimensional unitary matrices. Namely, one can give a *deterministic* rule for how the particle moves around—a differential equation—that still reproduces the Born rule at every instant in time, provided only that it reproduces the Born rule at any *one* time. More poetically, "God needs to use a random-number generator to initialize the hidden variables at the beginning of time"—say, at the Big Bang—but afterwards, they just follow the differential equation. And furthermore, while the choice of differential equation isn't quite unique, in simple scenarios (like a particle moving around in space) there's one choice that seems *better*, simpler, more motivated than the rest.

However, in thinking through the implications of Bohmian mechanics, Bohm and others noticed lots of weird things. It looks very elegant with just one particle, but new issues arise when there are two entangled particles. Bohmian mechanics says that you need to give a definite position for both particles, but people noticed that acting on Alice's particle would *instantaneously* change the Bohmian position of Bob's particle, however far away the particles were—even while Bob's *density matrix* remained unchanged because of the No-Communication Theorem.

While unsettling, this still wouldn't be useful for faster-than-light communication, since the Bohmian hidden variables are explicitly designed to have no measurable effects, over and above the effects we'd predict using the quantum state itself.

When Bohm proposed his interpretation, he was super eager for Einstein to accept it, but Einstein didn't really go for it, probably because of this non-locality problem.

What Einstein really wanted (in modern terms), is a…

Local Hidden Variable Theory

where hidden variables not only exist, but can be localized to specific points in space, and are only influenced by things happening close to them.

For example, imagine that when an entangled pair $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is created, the qubits secretly flip a coin and decide, "if anyone measures us in the $\{|0\rangle, |1\rangle\}$ basis, let's both be 0." More broadly, imagine that they agree in advance on such answers for all questions that could be asked (i.e., all bases in which they could possibly be measured), and that each qubit carries around its own local copy of the answers.

This is not Bohmian mechanics. In fact, around 1963 John Bell wrote a paper that drew attention to the non-local character of Bohmian mechanics. Bell remarked that it would be interesting to prove that *all* hidden variable theories must be non-local: that this isn't just a defect of Bohm's proposal, but inherent to what Bohm was trying to do. The paper has a footnote saying that as the paper was going to press, such a proof was found. This was the first announcement of one of the most famous discoveries ever made about quantum mechanics, what we now call

Bell's Inequality / Bell's Theorem

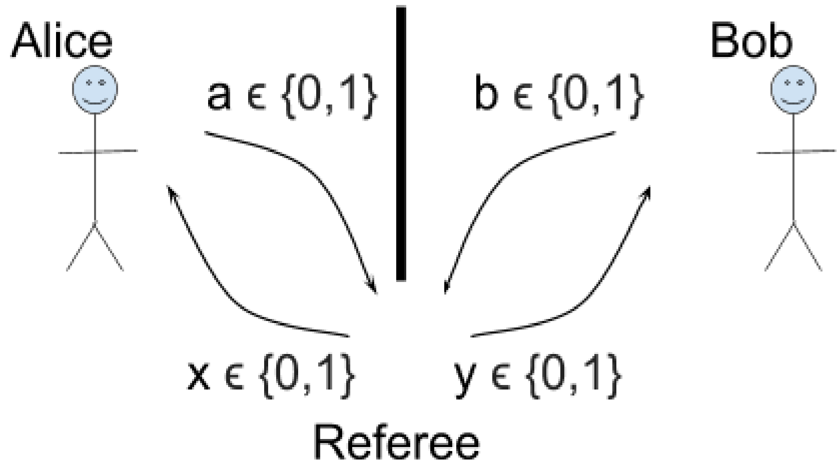
Einstein and others had already touched on the idea of local hidden variables in their philosophical debates in the 1930s. But Bell was the first to ask: do local hidden variables have any *empirical consequences* that disagree with the predictions of quantum mechanics? Is there an actual experiment that could rule out the possibility of local hidden variables?

Bell came up with such an experiment. We'll describe it differently from how Bell did—more computer science-y—as a game with two cooperating players named (what else?) Alice and Bob, where the win probability can be improved through shared entanglement. It's called…

The CHSH Game

named after four people (Clauser, Horne, Shimony, and Holt) who in 1969 wrote a paper saying "*this* is how to think about what Bell did." The game itself doesn't involve quantum mechanics, but quantum mechanics can help us win it.

The CHSH game could be seen as a precursor to quantum computing, in that it's one of the first cases where people looked to see which information processing tasks quantum mechanics helps us solve better—and where they enforced a conceptual separation between the task itself (which is classical), and the strategy to solve it (which can be quantum).



The idea is that Alice and Bob are placed in separate rooms, and are each given a challenge bit (*x* and *y*, respectively) by a referee. The challenge bits are chosen uniformly at random, and independently of each other. Then Alice sends an answer bit *a* back to the referee, and Bob sends back an answer bit *b*.

Alice and Bob "win" the game iff $a + b = xy \pmod 2$

So if either x or y is 0: a and b should be equal

But if $x = y = 1$: a and b should be different

Alice and Bob are allowed to agree on a strategy in advance, and to share random bits.

The classical strategy to maximize winning probability is simply that Alice and Bob always send the referee $a = b = 0$, regardless of what x and y are. In this case, Alice and Bob win 75% of the time, losing only if x and y are both 1.

To prove that this is optimal, the first step is to notice that, without loss of generality, Alice and Bob's strategy can be assumed to be deterministic (i.e., to involve no random bits besides x and y themselves). For any probabilistic strategy is just a mixture of deterministic ones—but then the win probability is just the average over all the strategies, so there must be *some* deterministic strategy in the mixture that does at least as well as the average. (This is called a *convexity argument*.)

So we can treat Alice's output bit, a, as a function of her input bit x, and Bob's output bit b as a function of his input bit y. And then we need the equation

$$a(x) + b(y) = xy \pmod 2$$

OK, but you can easily check by enumerating cases that this equation can't possibly hold for all 4 values of x and y! At best it can hold for 3 of the 4 values, which is exactly what the trivial strategy above gets.

The Bell Inequality, in this framework, is just the slightly-boring statement that we proved above: namely, that the maximum classical win probability in the CHSH game is 75%.

Bell noticed an additional fact though. Namely, if Alice and Bob had a pre-shared Bell pair, $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$, *then* there's a better strategy. In that case, in fact, their maximum win probability is

$$\cos^2(\frac{\pi}{8}) = \frac{1 + \sqrt{\frac{1}{2}}}{2} \sim 85\%$$

Why? Tune in next time to find out!