6.S899 Project Report
Tensor Networks, Quantum Error Correction, and AdS/CFT

John Napp

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1 Introduction

In recent years, the high energy physics community has become increasingly interested in entanglement, especially since it seems that, in some sense, spacetime itself may arise from entanglement [1]. Since dealing with entanglement is a strong point of quantum information theory, it's perhaps not surprising that quantum information is working its way into some work in this area, including work in the context of the AdS/CFT correspondence, a duality between quantum theories of gravity in $d + 1$ dimensions and conformal field theories in d dimensions. In this report, I survey at a very schematic level some recent work which lies in the intersection of quantum information theory (broadly speaking) and the AdS/CFT correspondence, especially in relation to tensor network models of holography. For completion, I won't assume background which wasn't covered in class, and so I briefly review relevant features of AdS/CFT and tensor networks.

2 AdS/CFT

As mentioned above, the AdS/CFT correspondence is a correspondence between quantum gravity in AdS on the one hand and a CFT in one dimension lower on the other, discovered by Juan Maldacena in the context of string theory in late 1997 [2]. The correspondence effectively provides a complex dictionary between states and operators on the two sides of the duality.

One result we will need is the AdS-Rindler reconstruction [3], which tells us where we can reconstruct local bulk operators on the boundary theory. For our elementary purposes in this report, it suffices to think of a single slice of low-dimensional AdS space represented by a disk. Schematically, the quantum gravity theory lives on the interior of the disk, and the conformal field theory lives on the boundary. Consider some connected region A of the boundary, and let γ be the geodesic through the bulk connecting the ends of this boundary, as in Figure 1. The AdS-Rindler reconstruction tells us that any bulk local operator inside

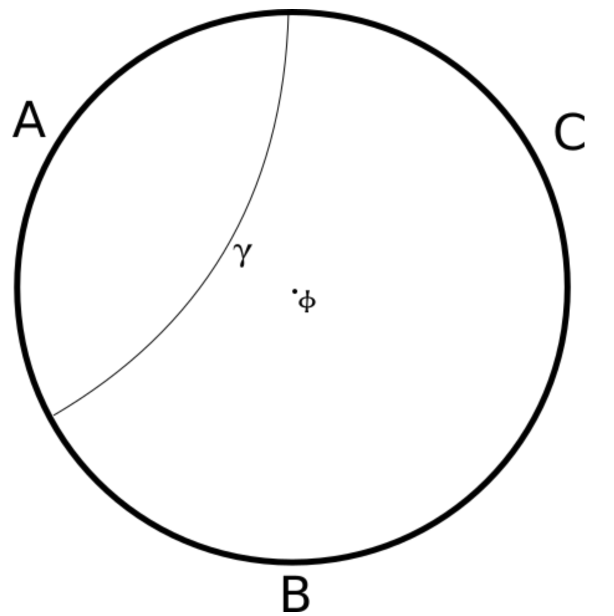

Figure 1: Low-dimensional depiction of bulk and boundary. A quantum theory of gravity lives on the interior in AdS space, and a dual CFT lives on the boundary. The boundary is split into three connected regions: A, B and C. γ is the minimal area surface (here, a geodesic) with boundary ∂A. ϕ is a local bulk operator.

the region bounded by A and γ can be reconstructed in the boundary theory as an operator acting nontrivially only on the subregion A.

Another important result for our purposes is the Ryu-Takayanagi formula [4]. Again considering the simplistic disk depiction of AdS/CFT above, a simplified version of the formula is

$$S_A = \frac{|\gamma|}{4G}$$

where S_A is the entanglement entropy of region A, $|\gamma|$ is the length of γ, and G is just a constant.

3 Tensor Networks

Tensor networks efficiently describe quantum many-body states, with pictorial representations reminiscent of Feynman diagrams. More specifically, tensor networks can neatly describe quantum entanglement in a system, when in general, quantum entanglement is extremely complicated to describe. Consider some multipartite quantum state

$$|\varphi\rangle = \sum_{i_1,\ldots,i_n} C_{i_1,\ldots,i_n} |i_1\rangle \otimes \cdots \otimes |i_n\rangle$$

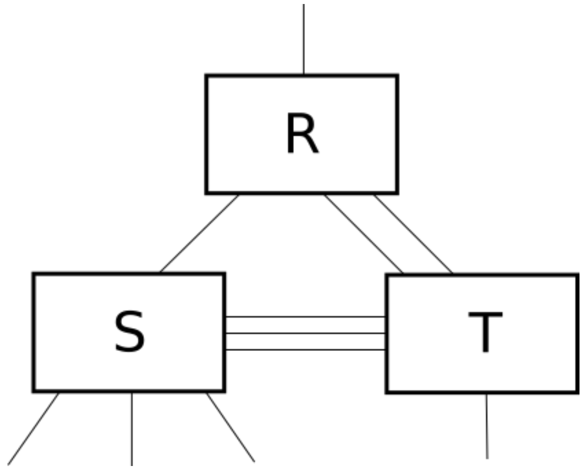

Figure 2: A simple tensor network consisting of three tensors R, S, and T. In this network, three indices are contracted between S and T, one between S and R, and two between R and T. After the contractions are performed, a single tensor remains with five uncontracted indices, and hence this network can describe a multipartite state on, for instance, five d-level subsystems.

We can assume that the subsystems have d levels, so each index in the sum runs from 1 to d. Note that we can think of the coefficients in this expression as elements of the tensor C, where now we are effectively thinking of C as a generalization of a matrix.

We can pictorially represent $|\varphi\rangle$ as a box representing C, with n legs coming out, representing the n d-level degrees of freedom the tensor is contracted with. We can also contract indices of different tensors together, to form new tensors. Diagrammatically, this is represented by legs connecting different tensors, as in Figure 2. For a recent review of tensor networks, see [5].

4 Tensor Networks and AdS/CFT

MERA is a family of tensor networks which is generally used to find the ground states of certain critical quantum systems in a computationally efficient manner. It was proposed by Vidal in 2006 in [6]. A connection between quantum information and holography (and the first instance to my knowledge of tensor networks being considered in the context of holography) came in 2009 when Brian Swingle noticed a connection between AdS/CFT and MERA [7]. He later elaborated on these ideas in [8]. This connection has been referred to as the "AdS/MERA" correspondence. While a proper summary of this connection would require more technical details than would be appropriate for this report, we can summarize some general ideas here. MERA proceeds in a repeating sequence of two steps. Thinking about a critial 1D spin chain with long-range entanglement, MERA proceeds by first applying "disentanglers" to adjacent pairs of spins. These are unitary transformations that serve to remove some short-range entanglement. Next, maps are applied to each such pair that coarse-grain. For our purposes, we can think of these as maps $V \times V \to V$ which effectively halve the number of spins in the chain (other coarse-graining schemes are possible). Running in the other direction, these maps $V \to V \times V$ are isometries (inner product preserving). We can repeat this sequence of two steps as many times as we want, reducing the number of degrees of freedom each time. In the end, we will be left with a coarse-grained version of our original state. We can alternatively run the MERA network in reverse, taking a low dimensional system to a much larger system with a pattern of long-range entanglement encoded by the tensors in the network.

Swingle noticed that a MERA network can be interpreted as generating an extra dimension with hyperbolic geometry. As an explicit example, consider a MERA network corresponding to a one-dimensional critical spin chain, such that in each coarse-graining step, the number of spins is cut in half. On the other hand, consider anti-de Sitter space with two spatial dimensions, with metric given by

$$ds^2 = \frac{l_{AdS}^2}{r^2}(-dt^2 + dx^2 + dr^2)$$

where l_{AdS} is a constant, t is the time coordinate, and x and r are spatial coordinates. Consider two points $x = u$ and $x = v$ on the boundary at $r = 0$ of this space separated by a distance R. The boundary at $r = 0$ is infinitely far away, but we can regulate it with some scale a (that is, we consider $r = a$ in the limit $a \to 0$). One can calculate the geodesic between u and v to be a semicircular path through the bulk with length $2l_{AdS} \ln(R/a)$. Also, the distance between two points (x_1, r) and (x_2, r) in the bulk traveling along a path of constant r is $l_{AdS}|x_1 - x_2|/r$.

Turning back to the MERA network (depicted in Figure 3), call the spatial dimension x'. Consider two sites s_1 and s_2 at the top of the network (that is, in the original state without any coarse-graining), and define the distance between them to be R/a. Now think of MERA as generating a second, vertical spatial dimension called r'. Now put a "taxicab metric" on the MERA graph. That is, a move from one site in the graph to another is one unit of distance. The allowed moves are either horizontal to an adjacent spin or up or down the network in the direction of coarse-graining. So, at each level of coarse-graining, one can move from a spin to an adjacent spin, and one can also move from a spin to the coarse-grained version of the spin (or vice versa). Note that the shortest path between sites s_1 and s_2 is the path that goes all the way down in the r' direction as far as possible, and then goes back up the network (a sort of discrete semicircle). The length of this path is $2\log_2(R/a)$, in correspondence with the result in the continuous case. We can think of a step in the coarse-graining direction as being associated with a distance of $u = \log_2(r/a)$. Then an increase of u by 1 corresponds to an increase of r by a factor of 2. Since half of the spins are removed at each step in the coarse-graining direction, the distance between two points in the x direction is halved in each u step. So if the original distance in the x' direction between the sites is R/a, then after n steps of coarse-graining, the distance between the

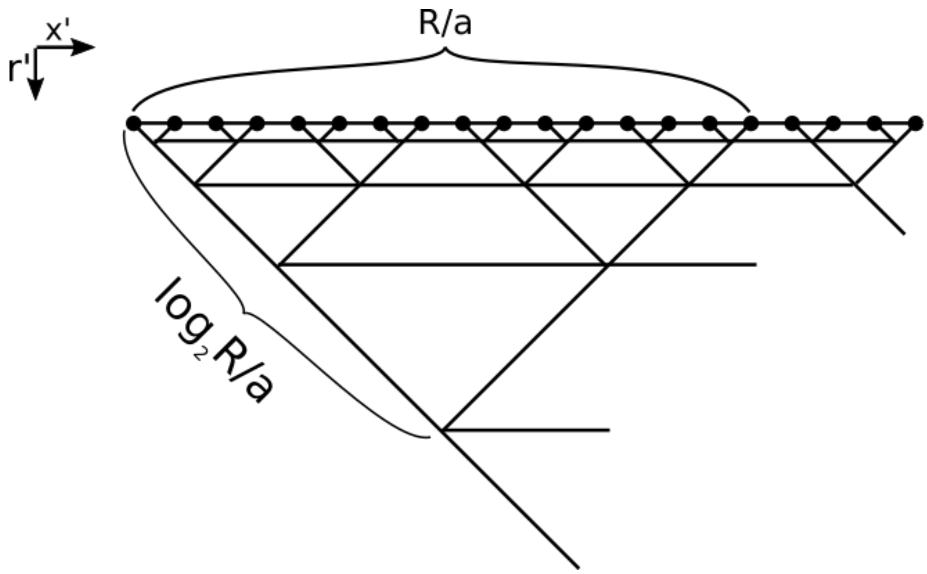

Figure 3: Schematic depiction of a MERA network, with disentangling gates suppressed. Only the course-graining is shown. One can think of the dots as representing physical spins, and moving down the network the system becomes increasingly coarse-grained. The length of a path in this network is essentially defined to be the number of edges comprising the path. In general, the shortest path between two dots travels down and then up the network, taking only "vertical" steps. This is analogous to the AdS case, in which the shortest path between two boundary points is a semicircle through the bulk. In this diagram, a is to be interpreted as a short-distance cutoff.

sites in the x' direction is $R/(a2^n) = R/r$, in correspondence with the continuous case.

The above is just a simple, intuitive example of how MERA can be thought of as generating a hyperbolic geometry. For detailed analyses of MERA in relation to geometry and holography, see [7], [8], [9], [10]. In particular, [10] is the most up-to-date account of the state-of-affairs in regards to the AdS/MERA correspondence. In this paper, the authors challenge MERA to reproduce certain features of AdS/CFT that one would expect should hold for a good correspondence. They find that MERA must obey certain constraints and that a strict AdS/MERA correspondence falls short of what's required in certain respects, but conclude that a looser definition of MERA may still work.

5 Quantum Error Correction and AdS/CFT

It was proposed in [12] that AdS/CFT might be realizing some form of quantum error-correction. It is easiest to understand this with a simple example. Consider a disk representing the bulk and boundary, and consider dividing the boundary of the disk into three equal-length sections, labeled A, B, and C (Figure 1). Now, consider some bulk local operator ϕ located in the center of the bulk. Note that from the AdS-Rindler reconstruction, ϕ should not be reconstructable on the boundary theory as an operator acting nontrivially only on region A, only on region B, or only on region C. However, we *do* expect ϕ to be reconstructable on AB, AC, and BC. But if the reconstruction of ϕ acts nontrivially only on AB, then it should commute with any operator acting nontrivially only on C. By similar reasoning, we conclude that the reconstruction of ϕ commutes with any operator acting only on A, on B, or on C. But then since it commutes with all local operators on the boundary, by a theorem from quantum field theory it must be proportional to the identity operator, which doesn't seem to make sense.

In [12], the authors proposed a way out of this puzzle by arguing that the reconstruction may yield different reconstructions of ϕ on different regions of the boundary, say $\tilde{\phi}_1$ on AB, $\tilde{\phi}_2$ on AC, and $\tilde{\phi}_3$ on BC, as long as these operators act equivalently on a certain low-energy subspace of the Hilbert space of the boundary conformal field theory, and hence AdS/CFT is doing some sort of quantum error correction. Specifically, in this case we can think of the situation as error correction on a tripartite system, in which we can perform a certain operation by acting on any two subsystems, but not on a single subsystem. [12] elaborates on this idea, and essentially argues that AdS/CFT realizes a quantum secret sharing scheme. One can imagine that some quantum information is living at a point in the bulk, and via the AdS/CFT dictionary that information is also encoded in the boundary theory. If some regions of the boundary are erased, one can still recover the bulk information as long as too much of the boundary is not erased. But once too much of the boundary is erased, it is impossible to even partially recover the bulk information.

6 Tensor Networks, Quantum Error Correction, and AdS/CFT

Following these connections between AdS/CFT, tensor networks, and quantum error correction, the authors of [13] constructed a family of quantum error correction codes based on tensor networks that serve as toy models for AdS/CFT. This work does not use MERA, but rather their tensor networks are based on so-called "perfect tensors". Consider a tensor $A_{i_1 \dots i_m}$. Note that we can take some subset $S \subset \{1, \dots, m\}$ of the indices, and consider A to describe a map from the degrees of freedom associated with S to the degrees of freedom associated with the complement of S. A is defined to be a perfect tensor if, for any such subset of the indices S with $|S| \leq \lfloor m/2 \rfloor$, the map induced by A from the degrees of freedom associated with S to the degrees of freedom associated with the complement of S is an isometry, meaning it preserves inner products. It is worthwhile to note that these perfect tensors are connected with other concepts in quantum information theory. For example, if we have a perfect tensor T with $2n$ indices, then the state associated with T is *absolutely maximally entangled*, meaning that any subsystem of n of its spins is maximally entangled with the complementary n spins. Furthermore, if we regard T as a map from one of its degrees of freedom to the complementary $2n-1$ degrees of freedom, then it gives

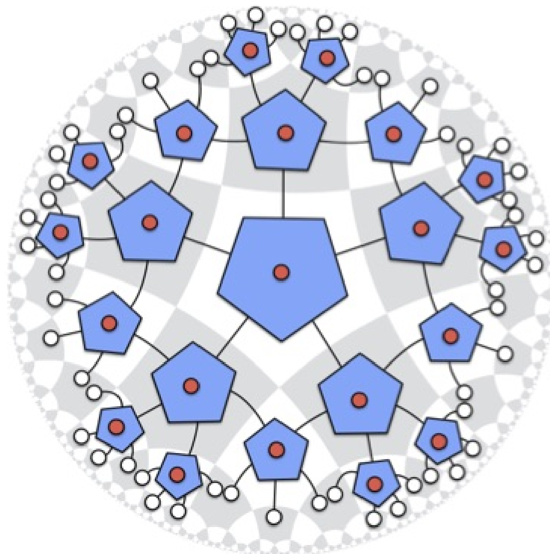

Figure 4: Holographic pentagon code. Image from [13]. The hyperbolic disk is tiled with pentagons, and then six-legged perfect tensors are placed on each tile. In this figure, the legs of each tensor are the five obvious legs, plus the red dot which represents a dangling bulk leg. One can think of the network as furnishing a map from the (logical) dangling bulk legs to the (physical) uncontracted boundary legs.

an encoding map from a logical spin to a block of $2n-1$ physical spins which is protected against the erasure of any $n-1$ physical spins.

These perfect tensors are the building blocks with which the authors construct families of "holographic codes". In summary, the hyperbolic disk is first tiled with polygons – say, pentagons. On each of these pentagons is placed a perfect tensor with six legs, as in Figure 4. Each tensor has one leg which is a "dangling bulk leg" and is not contracted. Tensors lying on the edge of the disk have some legs besides their dangling bulk legs which are uncontracted – these are boundary legs, and are associated with degrees of freedom living on the boundary of the disk. All other legs are contracted. Together, these tensors form a network which gives an exact mapping between bulk degrees of freedom associated with the dangling bulk legs, and boundary degrees of freedom. These mappings are quantum error correcting codes (stabilizer codes in some cases), explicitly realizing the quantum error correcting properties proposed in the work mentioned above. Properly constructed holographic codes realize discrete versions of properties of the AdS/CFT correspondence such as the Ryu-Takayanagi formula, AdS-Rindler reconstruction, and other features which I have not mentioned in this report. Another feature of perfect tensors is that operators can be "pushed" from one side of the tensor to the other side, which gives a notion of exact mappings of operators from bulk to boundary. One can imagine injecting an operator into a dangling bulk leg, after which it spreads via the network onto the bulk. There are different ways the injected operator can spread to the edge, explicitly realizing the non-uniqueness of the boundary reconstruction of local bulk operators.

7 Conclusion

The Ryu-Takayanagi formula shows us a deep connection between entanglement and geometry. We can describe entanglement with tensor networks, and recent work suggests that we can coax hyperbolic geometry out of them in some instances. The hope is that certain networks may serve as discrete models of AdS/CFT, especially since it seems that certain families may possess the error correction properties of AdS/CFT suggested in [12]. It will be exciting to see where this line of research goes in terms of understanding AdS/CFT, not to mention the inherent interest of these new "holographic codes" in quantum information theory.

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