SURVEY ON THE BOUNDS OF THE QUANTUM FAULT-TOLERANCE THRESHOLD

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1. Introduction

I first briefly summarize the threshold theorem and describe the motivations for tightening the bounds on the threshold quantum decoherence rate. I then go on to summarize and organize recent results regarding both the lower and the upper bounds for the threshold.

2. Motivation and Background

One of the major difficulties that experimental physicists and engineers face in attempting to build large scale quantum computers is dealing with decoherence (which is when the qubits inadvertently interact with their environment causing their states to collapse). Decoherence is almost inherently unavoidable, because quantum computation requires that the operator (who for this purpose is part of the environment) manipulate the qubits. Thus, the challenge of performing quantum computation under non-negligible rates of decoherence is very important and is investigated under the field of quantum fault tolerance.

Arguably the most central result to this field is known as the threshold theorem as proven by [Aharonov '96 FTQC], which can be summarized as follows:

Theorem 1. *There exists an error rate threshold $\eta_{th} > 0$ such that any ideal polynomially sized quantum circuit can be accurately simulated by a robust polynomial time quantum circuit that is resistant to any error rate $\eta < \eta_{th}$.*

Proof. Suppose each individual component of a fault tolerant quantum circuit with length T has an error rate less than a threshold η_{th} under concatenated error-correcting codes. Each level of encoding makes the effective error rate go from η to at most $\eta_{th}(\frac{\eta}{\eta_{th}})^2$. Therefore, after t levels of encoding the effective error rate becomes at most $\eta_{th}(\frac{\eta}{\eta_{th}})^{2^t}$. As a result, it requires only $O(\log\log T)$ levels of concatenation to achieve sufficient accuracy ($\eta_{\text{effective}} \ll \frac{1}{T}$), which requires only $O(\text{poly}\log T)$ extra qubits. $\square$

Note that this theorem implicitly makes some serious, non-trivial assumptions about quantum computation. Since qubits that are just waiting are still prone to error and quantum error correction requires fresh ancilla qubits, the theorem assumes that fresh ancilla qubits can be generated when needed rather than being created at the beginning of the computation. Similarly, the theorem assumes that quantum computations can be done in parallel, because otherwise we run in to the same problem of needing to waste resources error correcting for qubits that are not undergoing any calculations.

Despite these assumptions, this result triggered a wide range of research trying to calculate exact values for the constant η_{th} under various noise models and assumptions. However, the computed values η_{th} are still far from being precisely discovered. Computing progressively tighter bounds for η_{th} under varying assumptions has had significant implications on the realistic feasibility of large scale quantum computers. Lower bounds $\eta_{low} \leq \eta_{th}$ that were calculated with reasonable assumptions serve as goals for engineers in regards to how low to reduce the rate of decoherence. Conversely, upper bounds $\eta_{high} \geq \eta_{th}$ that were also calculated with reasonable assumptions serve as evidence to the feasibility (or infeasibility) of large scale quantum computing. Calculating tight values for both of these bounds has proven to be extremely difficult, due to the complications with choosing assumptions and how best to model the noise.

3. Lower Bound Results

In the past fifteen years, there has been a lot of progress in discovering lower bounds for η_{th}. These results can be divided into two categories: rigorous proofs that certain schemes are robust to particular error rates and estimates based on guessed assumptions and numerical simulation. As one might expect, the rigorous proofs have produced threshold values that are magnitudes lower than the best threshold estimates.

[Aharonov '96 FTQC], along with a few others, started the search for a lower bound in the late 1990's with an estimated value of 10^{-6}. A few years later, [Aliferis '05] rigorously proved a lower bound of 2.73×10^{-5}, by analyzing the fault tolerance of concatenated 7-qubit error correction codes. Although this threshold is still far too low for present day, practical purposes, concatenated 7-qubit codes are relatively easy to implement compared to other quantum error correction codes.

This rigorously proved lower bound was later improved by an order of magnitude to 1.9×10^{-4} by taking advantage of gauge sub-systems in 25-qubit Bacon-Shor codes [Alferis '06]. While this is a significantly better threshold than that of using 7-qubit error correction code, the Bacon-Shore code appears to be much more difficult to actually implement, because robustness is achieved by taking advantage of topology rather than concatenated error correction codes. Last year, [Paetznick] took advantage of 23-qubit Golay codes to help in the preparation of ancilla qubits, resulting in a scheme that had reasonable amounts of overhead and was rigorously proven to be robust to error rates less than 1.32×10^{-3}.

Starting a new direction, [Knill] introduced the concept of using post-selected computing to calculate lower bounds for the threshold, and produced a revolutionary estimated lower bound of 10^{-2}. Essentially, post-selected computation uses concatenating error detecting codes, as opposed to the error correction codes, to help in creating ancilla states. As the ancilla states are created, the ones which are detected to have errors are thrown out, allowing for relatively high fidelity ancilla qubits. [Aliferis '07] would later rigorously prove a lower bound for post-selection at a less impressive 1.04×10^{-3}. While this still improves the rigorously proven lower bound by another order of magnitude, it should be noted that the amount of resources necessary for post-selection appears to be prohibitively expensive.

It is important to note that all of the previously mentioned lower bounds have implicitly assumed that there are no geometric constraints (i.e. that it is possible to perform quantum operations between qubits regardless of physical distance) and most (with the exception of [Aliferis '05]) assume that there is negligible amounts of error correlation. Recently there has been a lot of work to find lower bounds for the threshold without these assumptions.

For example, [Stephens] proved a lower bound threshold of approximately 10^{-5} for computations using the 7-qubit Steane codes, while limiting interaction between nearest neighbor one dimensional systems. Similarly, for two dimensional systems, [Spedalieri] proved a lower bound threshold that is 2.02×10^{-5}.

Started in 1997 by [Dennis], using surface code error correction appears to have become the new main direction in quantum error correction. By taking advantage of physical properties of quantum mechanics, topological quantum computers are inherently more stable and less vulnerable to decoherence. Very recently, [Yao] and [Wang] demonstrated schemes that yield thresholds of $\approx 10^{-2}$, which is extremely promising. Even more remarkably, as recently as a few months ago, [Wootton] has presented an error correcting surface code that was efficiently robust to an error rate up to 1.85×10^{-1} under a relatively specific, yet hopefully, realistic model.

4. Upper Bound Results

While the majority of effort has been looking for lower bounds of the quantum fault-tolerance threshold, in recent years there have been a few important strides in tightening the threshold's upper bound. These upper bounds are very significant, because they demonstrate the maximum error rate tolerable for universal quantum computation, which allows experimentalists to better focus on lowering error rates.

Up until very recently, the upper bound threshold results have fallen into two categories. The first category, and arguably the stronger category, has been demonstrating that quantum circuits under particular error rates can be efficiently simulated by a classical computer. This category of results assumes, as do most theorists, that **BPP** $\neq$ **BQP**. In 1996, [Aharonov '96 PSDQC] produced the first upper bound of this kind with 9.7×10^{-1}. A few years later, [Harrow] improved this bound to 7.4×10^{-1}, by demonstrating that at this error rate it is impossible to produce entanglement with only one and two qubit gates. One thing to note, [Aharonov '96 PSDQC] used dephasing errors, while [Harrow] used depolarizing errors in their noise model. This was relatively recently improved to 4.53×10^{-1} by [Buhrman]. However, this improvement was made possible by using a more restrictive model that has "perfect" Clifford gates and having arbitrary one qubit gates being the only gates susceptible to error. [Plenio] was able to get considerably lower bounds (3.69×10^{-2}) using this method, but only applying to Clifford-based schemes.

The second category of results, demonstrate that the outputs of quantum circuits under particular error rates become indistinguishable from random after a logarithmic depth of computation. This category of results assumes, again as do most theorists, that $\mathbf{QNC^1} \neq$ **BQP**. It is important to note that [Cleve] proved that the QFT part of Shor's factoring algorithm, which is one of quantum computing's most powerful applications, can be implemented with logarithmic depth quantum circuits. Thus, the results from this category, while still important, are not as impressive as they would be otherwise. [Razborov] was the first of these results by showing that greater than logarithmic depth quantum computation is impossible with an error rate larger than $1 - \Theta(\frac{1}{k})$ where k is the max gate fan-in size allowed in the circuit (giving the strongest bound of 5×10^{-1} when $k = 2$).

After restricting the model a little bit, [Kempe] introduced the technique of Pauli basis decomposition to improve the bound to $1 - \Theta(\frac{1}{\sqrt{k}})$, which has a strongest bound of 3.57×10^{-1} when $k = 2$.

[Fern, Kay] have both recently produced upper bounds that are on the magnitude of 10^{-2}, but they appear to be analyzing relatively restrictive models and thus are not directly comparable with the previous results.

The author had trouble finding more recent results for upper bounds on the error rate threshold. It may be because as quantum error correction trends toward using surface codes, analysis for upper bounds has become even harder to do, due to the difficulties in rigorously analyzing surface codes.

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