Complexity of Quantum Field Theories (QFTs)

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Outline

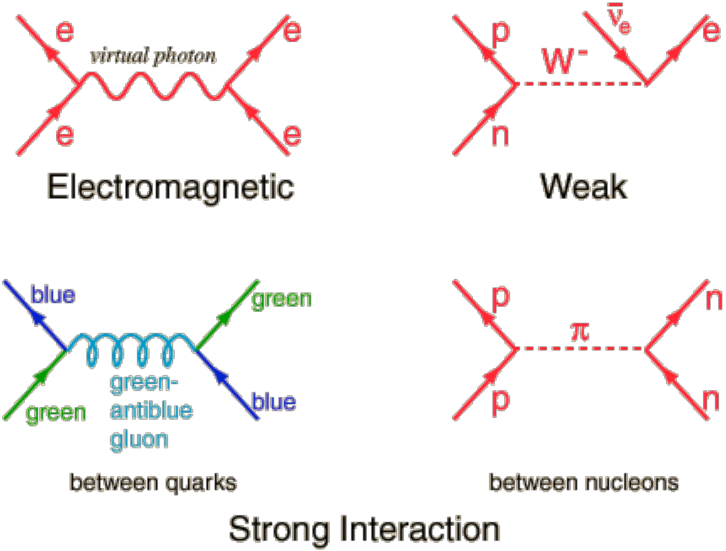
1. Introduction and Motivation
2. Intuition and Complications
3. Algorithm Overview

Outline

QFT: Marriage of QM and Special Relativity

Special Relativity: $E = m(c^2)$.
Infinitely Many Degrees of Freedom
$\Rightarrow$ QFT

Standard Model



Describing a QFT

Many different pictures of QM.

- Hamiltonian:

$$U = e^{-it\int d^3x\,\mathcal{H}}.$$

  This picks out a time coordinate – doesn't manifestly respect Lorentz symmetry

- Lagrangian:

$$Z = \int \mathcal{D}\phi e^{i\int d^4x\,\mathcal{L}}$$

$$\mathcal{L}_{QCD} = \overline{\psi}(i\not{D})\psi - \frac{1}{4}(F^i_{\mu\nu})^2 - m\overline{\psi}\psi$$

Approaches to QFT

- Perturbative – Taylor expand exponential. Asymptotic, not convergent. Requires small coupling
- Non-perturbative, exact – Only known for a few special cases, mostly in 2 spacetime dimensions
- Non-perturbative, numerical – Lattice. Gives static quantities (e.g., mass ratios), but not scattering amplitudes

No known efficient classical algorithm for scattering amplitudes, in general!

Outline

Non-Relativistic vs. Relativistic Computation

Classically, believed that special relativity gives no change in computation power. (Aaronson)

- Same degrees of freedom – positions, momenta
- Time dilation – relativity has more power?
- Cosmic speed limit – relativity has less power?

Is the same true, once we add in QM?

- QFT is more fundamental, so it should be able to simulate non-relativistic quantum computers in polynomial time
- Is the converse true?

Difficulties in Simulating a QFT with a Quantum Computer

- Initial state does not determine final state – probabilistic
- Depending on incoming momentum, could have arbitrarily many output particles
- Field can take on infinitely many values – can we impose max field cutoff and discretize without introducing too much error?
- Infinitely many degrees of freedom – can we impose UV and IR cutoffs without introducing too much error?

Outline

A Simple QFT – ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0}{4!}\phi^4, \quad \pi = \dot{\phi}$$

Future study: gauge theories, higher-spin fields, zero mass.

Algorithm

Spatial dimension d. Hamiltonian – easier to use quantum simulation. Put theory on spatial lattice $\Omega = a\mathbb{Z}_{\hat{L}}^d$ (with periodic bc); $\mathcal{V} = \hat{L}^d$ lattice sites

- Input: List of incoming momenta
- Output: List of outgoing momenta. ϕ mass non-zero $\Rightarrow$ $\#$ outoing particles is at most linear in incoming particles' center-of-mass energy. Repeated runs sample from probability distribution for possible results of scattering experiment
- Precision: Probability of an outcome, according to simulation, differs from true probability by no more than $\pm\epsilon$

Representing the Field – Single Lattice Site

$n_b = O(\log(\phi_{\max}/\delta_\phi))$. How to choose $\phi_{\max}, \delta_\phi$? Note: $\pi_{\max} \sim \frac{1}{a^d \delta_\phi}$. Say in $|\psi\rangle$, where $\langle\psi|H|\psi\rangle \leq E$. Unlikely for $|\phi(x)|$ to be much larger than $O(\sqrt{E})$.

$$\phi_{\max} = O\left(\sqrt{\frac{\mathcal{V}E}{a^d m_0^2 \epsilon}}\right), \quad \pi_{\max} = O\left(\sqrt{\frac{\mathcal{V}E}{\epsilon a^d}}\right)$$

$$n_b = O\left(\log\left(\frac{\mathcal{V}E}{m_0 \epsilon}\right)\right)$$

Algorithm

Suzuki-Trotter

1. Prepare the free vacuum (Kitaev and Webb)
2. Excite wavepackets.
3. Adiabatically turn on interaction.
4. Simulate Hamiltonian time evolution.
5. Adiabatically turn off interaction.
6. Measure occupation numbers of momentum modes.

Excite wavepackets

$$\Gamma = \frac{2\pi}{\hat{L}a}\mathbb{Z}_{\hat{L}}^{d}$$

$$\omega(\mathbf{p}) = \sqrt{m_0^2 + \frac{4}{a^2}\sum_{j=1}^{d}\sin^2\left(\frac{ap_j}{2}\right)}$$

$$a_{\mathbf{x}}^{\dagger} = \sum_{\mathbf{p}\in\Gamma} L^{-d} e^{-i\mathbf{p}\cdot\mathbf{x}} \sqrt{\frac{1}{2\omega(\mathbf{p})}} a_{\mathbf{p}}^{\dagger}$$

$$a_{\psi}^{\dagger} = \eta(\psi)\sum_{\mathbf{x}\in\Omega} a^d \psi(\mathbf{x}) a_{\mathbf{x}}^{\dagger}$$

$$H_{\psi} = a_{\psi}^{\dagger} \otimes |1\rangle\langle 0| + a_{\psi} \otimes |0\rangle\langle 1|$$

$$e^{-iH_{\psi}\pi/2}|0\rangle|0\rangle = -ia_{\psi}^{\dagger}|0\rangle|1\rangle$$

Adiabatically turn on interaction – How slowly do we go?

- Rate at which we turn on λ_0 depends on the physical mass.
- Weak coupling: can calculate perturbatively
- Strong coupling: slowly increase λ_0, estimating the mass at each value. Measure energy using phase estimation
- Route in parameter space may be circuitous – avoid "phase transition" where mass goes to 0

Representing the Field – IR Cutoff

Want V infinite – non-interacting, at first. Non-zero mass, so errors shrink exponentially with distance δ: $\epsilon \sim e^{-\delta/m} \Rightarrow V \sim n_{\text{in}} \log(1/\epsilon)$. Similarly, non-interacting at end, so $V \sim n_{\text{out}} \log(1/\epsilon)$. $p \ll 1/a$, so

$$\mathcal{V} \sim n_{\text{out}}^{d+1}$$

Representing the Field – UV Cutoff

$$\mathcal{L}_{\mathsf{eff}} = \mathcal{L}^{(0)} + \frac{c}{6!}\phi^6 + c'\phi^3\partial^2\phi + \frac{c''}{8!}\phi^8 + \dots$$

Dimensional analysis: later terms get larger power of a

Performance: Strong Coupling, $d = 1, 2$

	$\lambda_c - \lambda_0$	p	n_{out}
$d = 1$	$\left(\frac{1}{\lambda_c - \lambda_0}\right)^{8+o(1)}$	$p^{4+o(1)}$	$\tilde{O}(n_{\text{out}}^5)$
$d = 2$	$\left(\frac{1}{\lambda_c - \lambda_0}\right)^{5.04+o(1)}$	$p^{6+o(1)}$	$\tilde{O}(n_{\text{out}}^{7.128})$

Table: $f(n) = \tilde{O}(g(n))$ means $f(n) = O(g(n) \log^c(n))$ for some c

Performance: Weak Coupling

$$G_{\mathsf{weak}} \sim \begin{cases} \left(\frac{1}{\epsilon}\right)^{1.5+o(1)} & : d = 1 \\ \left(\frac{1}{\epsilon}\right)^{2.376+o(1)} & : d = 2 \\ \left(\frac{1}{\epsilon}\right)^{3.564+o(1)} & : d = 3 \end{cases}$$