Psi-Epistemic Theories: The Role of Symmetry

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Abstract

Formalizing an old desire of Einstein, "ψ-epistemic theories" try to reproduce the predictions of quantum mechanics, while viewing quantum states as ordinary probability distributions over underlying objects called "ontic states." Regardless of one's philosophical views about such theories, the question arises of whether one can cleanly rule them out, by proving no-go theorems analogous to the Bell Inequality. In the 1960s, Kochen and Specker (who first studied these theories) constructed an elegant ψ-epistemic theory for Hilbert space dimension $d = 2$, but also showed that any deterministic ψ-epistemic theory must be "measurement contextual" in dimensions 3 and higher. Last year, the topic attracted renewed attention, when Pusey, Barrett, and Rudolph (PBR) showed that any ψ-epistemic theory must "behave badly under tensor product." In this paper, we prove that even without the Kochen-Specker or PBR assumptions, there are no ψ-epistemic theories in dimensions $d \geq 3$ that satisfy two reasonable conditions: (1) symmetry under unitary transformations, and (2) "maximum nontriviality" (meaning that the probability distributions corresponding to any two non-orthogonal states overlap). The proof of this result, in the general case, uses some measure theory and differential geometry. On the other hand, we also show the surprising result that *without* the symmetry restriction, one can construct maximally-nontrivial ψ-epistemic theories in every finite dimension d.

1 Introduction

Debate has raged for almost a century about the interpretation of the quantum state. Although a quantum state evolves in a unitary and deterministic manner according to the Schrödinger equation, measurement is a probabilistic process in which the state is postulated to collapse to a single eigenstate. This is often viewed as an unnatural and poorly-understood process.

ψ-epistemic theories have been proposed as alternatives to standard quantum mechanics. In these theories, a quantum state merely represents probabilistic information about a "real, underlying" physical state (called the *ontic state*). Perhaps not surprisingly, several no-go theorems have been proven that strongly constrain the ability of ψ-epistemic theories to reproduce the predictions of standard quantum mechanics. Most famously, the Bell inequality [1]—while not usually seen as a result about ψ-epistemic theories—showed that no such theory can account for the results of all possible measurements on an entangled state in a "factorizable" way (i.e., so that the ontic state has a separate component for each qubit, and measurements of a given qubit only reveal information about that qubit's component of the ontic state). Also, the Kochen-Specker theorem

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[2] showed that in Hilbert space dimensions $d \geq 3$, no ψ-epistemic theory can be both deterministic and "noncontextual" (meaning that whether an eigenstate ψ gets returned as a measurement outcome is independent of which *other* states are also in the measurement basis). More recently, the Pusey-Barrett-Rudolph (PBR) theorem [3] showed that nontrivial ψ-epistemic theories are inconsistent, if the ontic distribution for a product state $|\psi\rangle \otimes |\phi\rangle$ is simply the tensor product of the ontic distribution for $|\psi\rangle$ with the ontic distribution for $|\phi\rangle$. Even more recently, papers by Maroney [4] and Leifer and Maroney [5] prove the impossibility of a "maximally ψ-epistemic theory," in which the overlap of the ontic distributions for all non-orthogonal states fully accounts for the uncertainty in distinguishing them via measurements.

In this paper, we study what happens if one drops the Bell, Kochen-Specker, and PBR assumptions, and merely asks for a ψ-epistemic theory in which the ontic distributions overlap for all non-orthogonal states.

Formally, a ψ-epistemic theory in d dimensions specifies:

1. A measurable space Λ, called the *ontic space* (the elements $\lambda \in \Lambda$ are then the ontic states).

2. A function mapping each quantum state $|\psi\rangle \in H_d$ to a probability measure μ_ψ over Λ, where H_d is the Hilbert space in d dimensions.

3. For each orthonormal measurement basis $M = \{\phi_1, \ldots, \phi_d\}$, a set of d *response functions* $\{\xi_{k,M}(\lambda) \in [0,1]\}$, which give the probability that an ontic state λ would produce a measurement outcome ϕ_k.

The response functions must satisfy the following two conditions:

$$\int_\Lambda \xi_{k,M}(\lambda)\, \mu_\psi(\lambda)\, d\lambda = |\langle \phi_k | \psi \rangle|^2, \tag{1}$$

$$\sum_{i=1}^{d} \xi_{i,M}(\lambda) = 1 \ \forall \lambda, M. \tag{2}$$

Here Equation (1) says that the ψ-epistemic theory perfectly reproduces the predictions of quantum mechanics (i.e., the Born rule). Meanwhile, Equation (2) says that the probabilities of the possible measurement outcomes must always sum to 1, even when ontic states are considered individually (rather than as elements of probability distributions). Note that Equations (1) and (2) are logically independent of each other.[1]

The conditions above can easily be satisfied by setting $\Lambda = \mathbb{CP}^{d-1}$, the complex projective space consisting of unit vectors in H_d up to an arbitrary phase, and $\mu_\psi(\lambda) = \delta(\lambda - \psi)$, where δ is the Dirac delta function, and $\xi_{k,M}(\lambda) = |\langle \phi_k | \lambda \rangle|^2$. But that simply gives an uninteresting restatement of quantum mechanics, since the μ_ψ's for different ψ's have disjoint supports.[2] Thus, let $\mathrm{Supp}(\mu_\psi) \subseteq \Lambda$ be the support of μ_ψ. Then we call a ψ-epistemic theory *nontrivial* if there exist $\psi \neq \phi$ such that μ_ψ and μ_ϕ have total variation distance less than 1, i.e.

$$\frac{1}{2} \int_\Lambda |\mu_\psi(\lambda) - \mu_\phi(\lambda)|\, d\lambda < 1. \tag{3}$$

[1] Also, we call a ψ-epistemic theory *deterministic* if the response functions take values only in $\{0,1\}$. The Kochen-Specker theorem then states that, in dimensions $d \geq 3$, any deterministic theory must have response functions that depend nontrivially on M.

[2] Indeed, some authors would not even call this trivial theory "ψ-epistemic"; they would insist on calling it "ψ-ontic" instead. In this paper, we adopt a uniform definition of ψ-epistemic theories, but then distinguish between trivial and nontrivial such theories.

If μ_ψ and μ_ϕ have total variation distance less than 1, then we say that $|\psi\rangle$ and $|\phi\rangle$ have "nontrivial overlap". Otherwise we say they have "trivial overlap". Note that it's possible for $|\psi\rangle$ and $|\phi\rangle$ to have trivial overlap even if μ_ψ and μ_ϕ have intersecting supports (this can happen if $\mathrm{Supp}(\mu_\psi) \cap \mathrm{Supp}(\mu_\phi)$ has measure 0).

Note also that if $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, then if we set $|\phi_1\rangle = |\psi\rangle$ and $|\phi_2\rangle = |\phi\rangle$, the conditions $|\langle\phi_1|\psi\rangle| = |\langle\phi_2|\phi\rangle| = 1$ and $|\langle\phi_2|\psi\rangle| = |\langle\phi_1|\phi\rangle| = 0$ imply that μ_ψ and μ_ϕ have trivial overlap. Hence, we call a theory *maximally nontrivial* if the overlap is *only* trivial for orthogonal states: that is, if all non-orthogonal states $|\psi\rangle, |\phi\rangle$ have nontrivial overlap.

In a maximally nontrivial theory, some of the uncertainty of quantum measurement is explained by the overlap between the distributions corresponding to non-orthogonal states. Recently Maroney [4] and Leifer and Maroney [5] showed that it is impossible to have a "maximally ψ-epistemic theory" in which *all* of the uncertainty is explained by the overlap of distributions. Specifically, they require that, for all quantum states $|\psi\rangle, |\phi\rangle$,

$$\int_{\mathrm{Supp}(\mu_\phi)} \mu_\psi(\lambda) d\lambda = |\langle\phi|\psi\rangle|^2. \tag{4}$$

Here we are asking for a much weaker condition, in which only *some* of the uncertainty in measurement statistics is explained by the overlap of distributions, and we do not impose any conditions on the amount of overlap.

Another property that we might like a ψ-epistemic theory to satisfy is *symmetry*. Namely, we call a ψ-epistemic theory *symmetric* if $\Lambda = \mathbb{CP}^{d-1}$ and the probability distribution $\mu_\psi(\lambda)$ is symmetric under unitary transformations that fix $|\psi\rangle$—or equivalently, if μ_ψ is a function f_ψ only of $|\langle\psi|\lambda\rangle|$. We stress that this function is allowed to be different for different ψ's: symmetry only applies to each μ_ψ individually. This makes our no-go theorem for symmetric theories stronger. If additionally μ_ψ is a *fixed* function f only of $|\langle\psi|\lambda\rangle|$, then we call the theory *strongly symmetric*. Note that, if a theory is strongly symmetric, then in order to apply a unitary U to a state $|\psi\rangle$, one can simply apply U to the ontic states. So strongly symmetric theories have a clear motivation: namely, they allow us to keep the Schrödinger equation as the time evolution of our system.

A similar notion to symmetry was recently explored by Hardy [6] and Patra et al. [7]. Given a ψ-epistemic theory, it is natural to consider the action of unitaries on the ontic states $\lambda \in \Lambda$. Hardy and Patra et al. define such a theory to obey "ontic indifference" if for any unitary U such that $U|\psi\rangle = |\psi\rangle$, and any $\lambda \in \mathrm{Supp}(\mu_\psi)$, we have $U\lambda = \lambda$. They then show that no ψ-epistemic theories satisfying ontic indifference exist in dimensions $d \geq 2$. Note that symmetric theories and even strongly symmetric theories need not obey ontic indifference, since unitaries can act nontrivially on ontic states in $\mathrm{Supp}(\mu_\psi)$. So the result of Hardy and Patra et al. is incomparable with ours.

In dimension 2, there exists a strongly symmetric and maximally nontrivial theory found by Kochen and Specker [2]. In dimensions $d \geq 3$, Lewis et al. [8] found a nontrivial ψ-epistemic theory for all finite d. However, their theory is not symmetric and is far from being maximally nontrivial.

In this paper, we first give a construction of a maximally nontrivial ψ-epistemic theory for arbitrary d. Our theory builds on that of Lewis et al. [8], and was first constructed in a post on MathOverflow [9]. Unfortunately, this theory is rather unnatural and is not symmetric. We then prove that it is impossible to construct a maximally nontrivial theory that *is* symmetric, for Hilbert space dimensions $d \geq 3$. In other words, if we want maximally nontrivial theories in 3 or more dimensions, then we either need an ontic space Λ other than $\Lambda = \mathbb{CP}^{d-1}$, or else we need ontic distributions μ_ψ that "single out preferred directions in Hilbert space," depending on more than just the inner product $|\langle\psi|\lambda\rangle|$.

2 Nonsymmetric, Maximally Nontrivial Theory

By considering $\Lambda = \mathbb{CP}^{d-1} \times [0,1]$, Lewis et al. [8] found a deterministic, nontrivial ψ-epistemic theory for all finite d. They raised as an open problem whether a *maximally* nontrivial theory exists. In this section, we answer their question in the affirmative. Specifically, we first show that, for any two non-orthogonal states, we can construct a theory such that their probability distributions overlap. We then take a convex combination of such theories to obtain a maximally nontrivial theory.

Lemma 1. *Given any two non-orthogonal quantum states $|a\rangle, |b\rangle$, there exists a ψ-epistemic theory $T(a,b) = (\Lambda, \mu, \xi)$ such that μ_a and μ_b have nontrivial overlap. Moreover, for $T(a,b)$, there exists $\varepsilon > 0$ such that $\mu_{a'}$ and $\mu_{b'}$ have nontrivial overlap for all $|a'\rangle$, $|b'\rangle$ such that*

$$||a - a'||, ||b - b'|| < \varepsilon.$$

Proof. Our ontic state space will be $\Lambda = \mathbb{CP}^{d-1} \times [0,1]$. Given an orthonormal basis $M = \{\phi_1, \ldots, \phi_d\}$, we first sort the ϕ_i's in decreasing order of $\min(|\langle\phi_i|a\rangle|, |\langle\phi_i|b\rangle|)$. Then the outcome of measurement M on ontic state (λ, p) will be the smallest positive integer i such that

$$|\langle\phi_1|\lambda\rangle|^2 + \cdots + |\langle\phi_{i-1}|\lambda\rangle|^2 \leq p \leq |\langle\phi_1|\lambda\rangle|^2 + \cdots + |\langle\phi_i|\lambda\rangle|^2. \tag{5}$$

In other words, $\xi_{i,M}(|\lambda\rangle, p) = 1$ if i satisfies the above and no $j < i$ does, and is 0 otherwise. If we assume that $\mu_\psi(|\lambda\rangle, p) = \delta(|\lambda\rangle - |\psi\rangle)$ for all $p \in [0,1]$, then it can be verified that $T(a,b)$ is a valid ψ-epistemic theory, albeit so far a trivial one.

We now claim that there exists $\varepsilon > 0$ such that, for all orthonormal bases $M = \{\phi_1, \ldots, \phi_d\}$, there exists i such that $|\langle\phi_i|a\rangle| \geq \varepsilon$ and $|\langle\phi_i|b\rangle| \geq \varepsilon$. Indeed, by the triangle inequality, we can let $\varepsilon = |\langle a|b\rangle|/d$, and $\varepsilon > 0$ since $|\langle a|b\rangle| > 0$. This means that, for all measurements M and all $p \in [0, \varepsilon]$, the outcome is always $i = 1$ when M is applied to either of the ontic states $(|a\rangle, p)$ or $(|b\rangle, p)$.

Following Lewis et al. [8], we can "mix" the probability distributions μ_a and μ_b, or have them intersect in the region $p \in [0, \varepsilon]$, without affecting the Born rule statistics for any measurement. Explicitly, we can let

$$E_{a,b} = \{|a\rangle, |b\rangle\} \times [0, \varepsilon], \tag{6}$$

so that all $\lambda \in E_{a,b}$ give the same measurement outcome ϕ_1 for all measurements M. Then any probability assigned by μ_a or μ_b to states within $E_{a,b}$ can be redistributed over $E_{a,b}$ without changing the measurement statistics. Thus, we can define μ_a such that the weight it originally placed on $|a\rangle \times [0, \varepsilon]$ is now placed uniformly on $E_{a,b}$. More formally, we set

$$\mu_a(|\lambda\rangle, x) = \begin{cases} \delta(|\lambda\rangle - |a\rangle) & \text{if } x > \varepsilon \\ \varepsilon \mu_{E_{a,b}}(|\lambda\rangle, x) & \text{if } x \leq \varepsilon\,. \end{cases} \tag{7}$$

where $\mu_{E_{a,b}}$ is the uniform distribution over $E_{a,b}$. We similarly define μ_b. This then yields a theory with nontrivial overlap between $|a\rangle$ and $|b\rangle$.

Furthermore, suppose we have $|a'\rangle$, $|b'\rangle$, such that $||a - a'||, ||b - b'|| < \frac{\varepsilon}{2}$. Then by continuity, we can similarly mix the distributions $\mu_{a'}$ and $\mu_{b'}$, or have them intersect each other in the region $p \in [0, \frac{\varepsilon}{2}]$, without affecting any measurement outcome. Note that the procedure of sorting the basis vectors of M might cause the measurement outcome to change discontinuously. However, this is not a problem since the procedure depends only on $|a\rangle$ and $|b\rangle$, which are fixed, and hence occurs uniformly for all $|a'\rangle$ and $|b'\rangle$ defined as above. $\square$

Lemma 1 implies that for any two non-orthogonal states $|a\rangle$ and $|b\rangle$, we can construct a theory where $\mu_{a'}$ and $\mu_{b'}$ have nontrivial overlap for all $||a-a'||, ||b-b'|| < \varepsilon$, for some $\varepsilon > 0$. To obtain a maximally nontrivial theory, such that any two non-orthogonal vectors have probability distributions that overlap, we take a convex combination of such ψ-epistemic theories.

Given two ψ-epistemic theories $T_1 = (\Lambda_1, \mu_1, \xi_1)$ and $T_2 = (\Lambda_2, \mu_2, \xi_2)$ and a constant $c \in (0,1)$, we define the new theory $cT_1 + (1-c)T_2 = (\Lambda_c, \mu_c, \xi_c)$ by setting $\Lambda_c = (\Lambda_1 \times \{1\}) \cup (\Lambda_2 \times \{2\})$ and $\mu_c = c\mu_1 + (1-c)\mu_2$. For any $(\lambda, i) \in \Lambda_c$, we then define ξ_c to equal ξ_i on Λ_i.

The following is immediate from the definitions.

Lemma 2. *$cT_1 + (1-c)T_2$ is a ψ-epistemic theory. Furthermore, if T_1 mixes the probability distributions μ_ψ, μ_ϕ of two states $|a\rangle$ and $|b\rangle$, and T_2 mixes $\mu_{a'}$ and $\mu_{b'}$, then $cT_1 + (1-c)T_2$ mixes both pairs of distributions, assuming $c \notin \{0,1\}$.*

Note that the ontic state space of a convex combination of theories contains a copy of each of the original ontic spaces Λ_1 and Λ_2. If $\Lambda_1 = \Lambda_2$, it is natural to ask if we could get away with keeping only one copy of the ontic state space. Unfortunately the answer in general is no. Suppose that we let $\Lambda_c = \Lambda_1 = \Lambda_2$, let $\mu_c = c\mu_1 + (1-c)\mu_2$, and let $\xi_c = c\xi_1 + (1-c)\xi_2$. Then the probability of measuring outcome i under measurement M and ontic distribution μ_{c_ψ} is

$$\int_\Lambda \left(c\xi_{1_{i,M}}(\lambda) + (1-c)\xi_{2_{i,M}}(\lambda)\right) \left(c\mu_{1_\psi}(\lambda) + (1-c)\mu_{2_\psi}(\lambda)\right) \, d\lambda$$

which will not in general reproduce the Born rule due to unwanted cross terms. This is why it is necessary to keep two copies of the ontic state space when taking a convex combination of theories.

Using Lemmas 1 and 2, we now construct a maximally nontrivial ψ-epistemic theory. Let $T(a,b)$ be the theory returned by Lemma 1 given $|a\rangle, |b\rangle \in H_d$. Also, for all positive integers n, let A_n be a $1/n$-net for H_d, that is, a finite subset $A_n \subseteq H_d$ such that for all $|a\rangle \in H_d$, there exists $|a'\rangle \in A_n$ satisfying $||a-a'|| < 1/n$. By making small perturbations, we can ensure that $\langle a|b\rangle \neq 0$ for all $|a\rangle, |b\rangle \in A_n$. Then our theory T is defined as follows:

$$T = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{1}{|A_n|^2} \sum_{a,b \in A_n} T(a,b) \right). \tag{8}$$

(Of course, in place of $6/(\pi^2 n^2)$, we could have chosen any infinite sequence summing to unity.) This yields a maximally nontrivial theory, since it can be verified that μ_a and μ_b have nontrivial overlap for all non-orthogonal states $|a\rangle$ and $|b\rangle$. Note that the ontic space is now $\mathbb{CP}^{d-1} \times [0,1] \times \mathbb{N}$, which has the same cardinality as $\mathbb{CP}^{d-1}$. It is thus possible to map this theory into a theory that uses $\Lambda = \mathbb{CP}^{d-1}$ as its ontic space, using a bijection between the ontic spaces. However, it is clear that under such a bijection the theory becomes less symmetric: the quantum state $|a\rangle$ no longer has any association with the state $|a\rangle$ in the ontic space, and the measure is also very unnatural.

3 Nonexistence of Symmetric, Maximally Nontrivial Theories

We now turn to showing that it is impossible to construct a *symmetric* maximally nontrivial theory, in dimensions $d \geq 3$. Recall that a theory is called symmetric if

1. $\Lambda = \mathbb{CP}^{d-1}$, and
2. for any quantum state $|\psi\rangle$, the associated probability distribution μ_ψ is invariant under unitary transformations that preserve $|\psi\rangle$.

Specifically, if U is a unitary transformation such that $U|\psi\rangle = |\psi\rangle$, then we require that $\mu_{U\psi}(U\lambda) = \mu_\psi(\lambda)$. This implies that $\mu_\psi(\lambda)$ is a function only of $|\langle\psi|\lambda\rangle|^2$: that is,

$$\mu_\psi(\lambda) = f_\psi\left(|\langle\psi|\lambda\rangle|^2\right) \tag{9}$$

for some nonnegative function f_ψ. In other words, the probability measure μ_ψ associated with state ψ must be a measure ν_ψ on the unit interval which has been "stretched out" onto H_d over curves of constant $|\langle\psi|\lambda\rangle|^2$. If additionally we assume that for *any* U, $\mu_{U\psi}(U\lambda) = \mu_\psi(\lambda)$, or equivalently that $\mu_\psi(\lambda) = f\left(|\langle\psi|\lambda\rangle|^2\right)$ for some *fixed* nonnegative function f, the theory is called strongly symmetric.

In this section, we will first prove several facts about symmetric, maximally nontrivial theories in general. Using these facts, we will then show that no *strongly* symmetric, maximally nontrivial theory exists in dimension 3 or higher. Restricting to the strongly symmetric case will make the proof considerably easier. Later we will show how to generalize to the "merely" symmetric case.

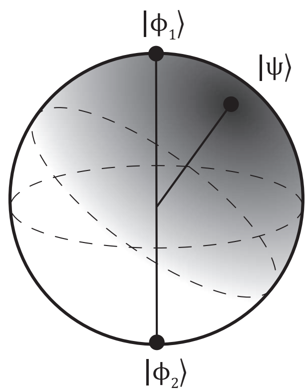

Figure 1: Diagram of maximally nontrivial theory in $d = 2$ on the Bloch sphere. The shaded region corresponds to Supp(μ_ψ).

As mentioned earlier, Kochen and Specker proved that a strongly symmetric, maximally nontrivial ψ-epistemic theory exists in dimension $d = 2$ [2]. In their theory, which is illustrated in Figure 1, the ontic space is $\Lambda = \mathbb{CP}^1$, and the response functions for a given basis $M = \{\phi_1, \phi_2\}$ are

$$\begin{aligned} \xi_{1,M}(\lambda) &= 1 \text{ if } |\langle\lambda|\phi_1\rangle| \geq |\langle\lambda|\phi_2\rangle| \text{ or } 0 \text{ otherwise,} && (10)\\ \xi_{2,M}(\lambda) &= 1 \text{ if } |\langle\lambda|\phi_2\rangle| \geq |\langle\lambda|\phi_1\rangle| \text{ or } 0 \text{ otherwise.} && (11)\end{aligned}$$

Hence the response functions are deterministic and partition the ontic space. Intuitively, the result of a measurement on any ontic state is the state in the measurement basis to which it is closest. For any quantum state $|\psi\rangle \in H_2$, the probability distribution over Λ is given by

$$\mu_\psi(\lambda) = \begin{cases} \frac{2}{\pi}\left(|\langle\lambda|\psi\rangle|^2 - \frac{1}{2}\right) & \text{if } |\langle\lambda|\psi\rangle|^2 > \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

It can readily be verified that this theory satisfies the conditions for a ψ-epistemic theory, and has the properties of being strongly symmetric and maximally nontrivial. It is also maximally ψ-epistemic in the sense described by Maroney [4] and Maroney and Leifer [5].

Given a measurement outcome ψ and a basis M containing ψ, we define the *nonzero set* Nonzero$(\xi_{\psi,M})$ to be the set of ontic states λ such that the response function $\xi_{\psi,M}(\lambda)$ gives a nonzero probability of returning ψ when M is applied:

$$\text{Nonzero}(\xi_{\psi,M}) = \{\lambda : \xi_{\psi,M}(\lambda) \neq 0\}. \tag{12}$$

Clearly in any ψ-epistemic theory, $\mathrm{Supp}(\mu_\psi) \subseteq \mathrm{Nonzero}(\xi_{\psi,M})$ for any measurement basis M that contains ψ, because the state $|\psi\rangle$ must return measurement outcome ψ with probability 1 for any such M. Harrigan and Rudolph [11] call a ψ-epistemic theory *deficient* if there exists a quantum state $|\psi\rangle$ and measurement basis M containing ψ such that

$$\mathrm{Supp}(\mu_\psi) \subsetneq \mathrm{Nonzero}(\xi_{\psi,M}). \tag{13}$$

In other words, a theory is deficient if there exists an ontic state λ such that λ has a nonzero probability of giving the measurement outcome corresponding to $|\psi\rangle$ for some M, even though $\lambda \notin \mathrm{Supp}(\mu_\psi)$. This can be thought of as a "one-sided friendship" between $|\psi\rangle$ and λ.

It was first pointed out by Rudolph [10], and later shown by Harrigan and Rudolph [11], that theories in dimension $d \geq 3$ must be deficient. In this section, we prove that as a result of deficiency, it is impossible to have a symmetric, maximally nontrivial theory with $d \geq 3$. We derive a contradiction by showing that if the theory is maximally nontrivial, then there exist orthogonal states $|\psi\rangle, |\phi\rangle$, and a measurement basis M containing $|\psi\rangle$, such that if $|\phi\rangle$ is measured, then the outcome $|\psi\rangle$ is returned with nonzero probability, contradicting the laws of quantum mechanics. We do this by choosing $|\psi\rangle$ orthogonal to $|\phi\rangle$ such that its associated response function $\xi_{\psi,M}$ has nonzero measure in the deficient region, and proving that there exists $|\phi\rangle$ such that $\mathrm{Supp}(\mu_\phi)$ overlaps with $\mathrm{Nonzero}(\xi_{\psi,M})$ in the deficient region. Thus the deficiency imposed by the symmetry assumption makes it impossible for a maximally nontrivial theory to exist in $d \geq 3$.

We start with a few preliminary results on symmetric, maximally nontrivial theories. As stated previously, we know that μ_ψ is generated by stretching a probability measure ν_ψ on the unit interval over H_d along spheres of constant $|\langle\psi|\lambda\rangle|^2$. By the Lebesgue decomposition theorem, ν_ψ can be written uniquely as a sum of two measures $\nu_{\psi,C}$ and $\nu_{\psi,S}$, where $\nu_{\psi,C}$ is absolutely continuous with respect to the Lebesgue measure over the unit interval, and $\nu_{\psi,S}$ is singular with respect to that measure. Here when we say $\nu_{\psi,C}$ is "absolutely continuous" with respect to the Lebesgue measure, we mean that it assigns zero measure to any set of Lebesgue measure zero. When we say $\nu_{\psi,S}$ is "singular," we mean that its support is confined to a set of Lebesgue measure zero. Similarly, μ_ψ can be decomposed into its absolutely continuous and singular parts $\mu_{\psi,C}$ and $\mu_{\psi,S}$, which are defined respectively from the components $\nu_{\psi,C}$ and $\nu_{\psi,S}$ of ν_ψ. By the Radon-Nikodym theorem, due to its absolute continuity $\nu_{\psi,C}$ has a probability density function $g_\psi(x)$ that is a function, not a pseudo-function or delta function. To simplify our analysis, first we will show that it is only necessary to look at the absolutely continuous part of the distribution.

Lemma 3. *For any distinct and non-orthogonal states $|\psi\rangle$, $|\phi\rangle$ in a symmetric, maximally nontrivial theory, $\mu_{\psi,C}$ and $\mu_{\phi,C}$ have nontrivial overlap.*

Proof. Let S_a denote the set of states $\lambda \in \Lambda$ with $|\langle\lambda|\psi\rangle|^2 = a$. If $a = 1$, then S_a is a single point with zero μ_ϕ measure. For $0 < a < 1$, S_a is a $(2d-3)$-sphere centered about ψ, and for $a = 0$ it is a $(2d-4)$-dimensional manifold diffeomorphic to $\mathbb{CP}^{d-1}$. In both of the latter cases, as ϕ, ψ are distinct non-orthogonal states, the distribution of $|\langle\lambda|\phi\rangle|^2$ for λ chosen uniformly on S_a is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. Therefore, the distribution of $|\langle\lambda|\phi\rangle|^2$ for $\lambda \in \Lambda$ chosen according to μ_ψ is absolutely continuous over $|\langle\lambda|\psi\rangle| < 1$.

By our symmetry condition $\mu_{\phi,S}$ is the product of a singular measure on [0,1], denoted $\nu_{\phi,S}$, and the uniform measure on rings of constant $|\langle\phi|\lambda\rangle|^2$. Since drawing λ from μ_ψ induces an absolutely continuous measure on $|\langle\phi|\lambda\rangle|^2$, then in particular μ_ψ has probability zero of producing a state λ with $|\langle\phi|\lambda\rangle|^2 \in \mathrm{Supp}(\nu_{\phi,S})$, because $\mathrm{Supp}(\nu_{\phi,S})$ is a set of measure zero. This implies that μ_ψ has probability zero of producing a state $\lambda \in \mathrm{Supp}(\mu_{\phi,S})$. Hence there is zero overlap between μ_ψ and $\mu_{\phi,S}$. In particular, $\mu_{\psi,C}$ and $\mu_{\psi,S}$ have zero overlap with $\mu_{\phi,S}$. Similarly, $\mu_{\phi,C}$ and $\mu_{\phi,S}$ have zero overlap with $\mu_{\psi,S}$.

This shows that the overlap between $\mu_{\phi,C}$ and $\mu_{\psi,C}$ equals that between μ_ϕ and μ_ψ, which is nonzero for maximally nontrivial theories. ☐

From now on, we will assume μ_ψ is generated only from the absolutely continuous part $\nu_{\psi,C}$, so that μ_ψ has as probability density function $f_\psi\left(|\langle\psi|\lambda\rangle|^2\right)$ where f_ψ is a function, not a pseudo-function. We can do this without loss of generality, as our proof will not depend on the normalization of the probability distributions, and will only use facts about the absolutely continuous components of the measures.

Next, let the distance between two states ψ and ϕ be defined by their scaled radial distance (also called the *Fubini-Study metric*):

$$||\psi - \phi|| = \frac{2}{\pi} \arccos\left(|\langle\psi|\phi\rangle|\right).$$

For any state $|\psi\rangle \in H_d$, with probability distribution $\mu_\psi(\lambda) = f_\psi(|\langle\psi|\lambda\rangle|^2)$, we define the *radius* of μ_ψ to be the distance between $|\psi\rangle$ and the furthest away state at which μ_ψ has substantial density:

$$r_\psi = \sup\left\{r : \forall\delta > 0 \int_{\lambda: r-\delta<||\psi-\lambda||<r} \mu_\psi(\lambda)\, d\lambda > 0\right\}. \tag{14}$$

Lemma 4. *For a symmetric theory, given any two states $|\psi\rangle, |\phi\rangle$, we have $||\psi - \phi|| \geq r_\psi + r_\phi$ if and only if $|\psi\rangle$ and $|\phi\rangle$ have trivial overlap.*

Proof. Suppose that $||\psi-\phi|| \geq r_\psi + r_\phi$ but $|\psi\rangle$ and $|\phi\rangle$ have nontrivial overlap. Then $\mathrm{Supp}(\mu_\psi) \cap \mathrm{Supp}(\mu_\phi)$ has nonzero measure, and for any λ in that set, the triangle inequality implies that $r_\psi + r_\phi \geq ||\psi - \lambda|| + ||\phi - \lambda|| \geq ||\psi - \phi||$. Thus r_ψ and r_ϕ satisfy $r_\psi + r_\phi = ||\psi - \phi||$, which is a contradiction since $||\psi - \lambda|| + ||\phi - \lambda|| = ||\psi - \phi||$ only on a set of measure zero.

Now suppose that $||\psi-\phi|| < r_\psi + r_\phi$. Consider λ_{int}, an ontic state which lies at the intersection of rings of radii r_ψ and r_ϕ about ψ and ϕ, respectively. In other words $||\psi - \lambda_{\text{int}}|| = r_\psi$ and $||\phi - \lambda_{\text{int}}|| = r_\phi$. Such a λ_{int} exists because $||\psi - \phi|| < r_\psi + r_\phi$. Then in the neighborhood of λ_{int}, we claim that μ_ψ and μ_ϕ have nontrivial overlap.

To show this, we will define a set B of positive measure, on which μ_ψ and μ_ϕ are "equivalent" to the Lebesgue measure, in the sense that if $S \subseteq B$ has positive Lebesgue measure, then S has positive measure under both μ_ψ and μ_ϕ. This implies that ψ and ϕ have nontrivial overlap on B.

By the symmetry condition, each μ_ψ is a product measure between a measure ν_ψ on $[0, 1]$ and a uniform measure on surfaces of constant $|\langle\psi|\lambda\rangle|$. Let u and v be the normal vectors to surfaces of constant $|\langle\psi|\lambda\rangle|$ and $|\langle\phi|\lambda\rangle|$ at $|\lambda_{\text{int}}\rangle$, respectively. Note that u (v) is equal to the tangent vector to the geodesic running from ψ (ϕ) to λ_{int} evaluated at λ_{int}. Since $||\psi - \phi|| < r_\psi + r_\phi$, these are distinct geodesics, so u and v are linearly independent.

Since $\mathbb{CP}^{d-1}$ is a smooth Riemannian manifold, u and v form a local coordinate system in the ε neighborhood of $|\alpha\rangle$, which we denote $N_\varepsilon(|\alpha\rangle)$. If we associate coordinates x_1, x_2 with u and v, the integral over $N_\varepsilon(|\alpha\rangle)$ can be parameterized as

$$\int g(x_1, x_2, y_1 \ldots y_{2d-4})\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1 \ldots \mathrm{d}y_{2d-4}$$

Here g is the square root of the metric, which is strictly positive in the neighborhood of $|\lambda_{\text{int}}\rangle$. Also, $\mathrm{d}x_i$ is the Lebesgue integral over the coordinate x_i, and the y_i are coordinates corresponding to the remaining $2d - 4$ dimensions of the space.

Now consider the set

$$B = N_\varepsilon(|\alpha\rangle) \cap \mathrm{Supp}(\mu_\psi) \cap \mathrm{Supp}(\mu_\phi)$$

Trivially μ_ψ and μ_ϕ are equivalent to the Lebesgue measure on B. Note that $\mathrm{Supp}(\mu_\psi)$ is a union of surfaces S_1 of constant $|\langle\lambda|\psi\rangle|$ which are perpendicular to u at α. If ε is sufficiently small these surfaces have negligible curvature, so they look like orthogonal hyperplanes in the x_1 coordinate system. Let ε_1 be the Lebesgue measure on $\mathrm{Supp}(\mu_\psi) \bigcap B$. Let S_2 and ε_2 be defined similarly for ϕ. If the surfaces S_1, S_2 had zero curvature, the Lebesgue measure of B would simply be the product of the measures $\varepsilon_1\varepsilon_2$, since x_1 and x_2 are orthogonal coordinates. Since the surfaces have slight curvature, and the coordinates x_i are not truly orthogonal, the above calculation has to be changed slightly. Specifically, for sufficiently small ε the Lebesgue measure of B can be approximated by $g\varepsilon_1\varepsilon_2$, where g is the square root of the metric at λ_{int}. This quantity is strictly positive since each $\varepsilon_i > 0$ by the definition of r, the metric g is strictly positive, and μ_ϕ and μ_ϕ are absolutely continuous with respect to the Lebesgue measure. Hence B has positive Lesbesgue measure.

□

Corollary 1. *If $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, then $r_\psi + r_\phi \leq 1$.*

Proof. If $\langle\psi|\phi\rangle = 0$, then $||\psi - \phi|| = 1$. Since any orthogonal $|\psi\rangle$ and $|\phi\rangle$ have trivial overlap, Lemma 4 implies that $r_\psi + r_\phi \leq 1$. □

Lemma 5. *Given any maximally nontrivial and symmetric theory in $d \geq 3$, for any state $|\psi\rangle \in H_d$, we have $r_\psi = \frac{1}{2}$.*

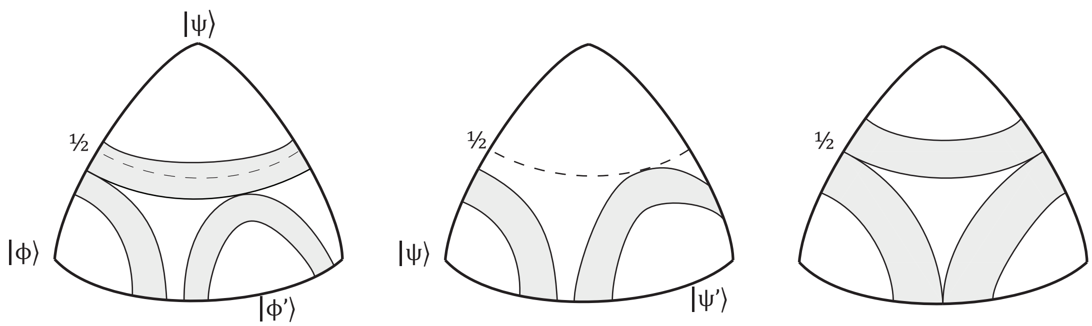

Figure 2: From left to right: pictorial representations of the proof that $r_\psi \leq \frac{1}{2}$ in dimension 3, the proof that $r_\psi = \frac{1}{2}$, and the form of the μ_ψ's that we ultimately deduce (with $r_\psi = \frac{1}{2}$ for all $|\psi\rangle \in H_d$). The shaded regions are the supports of the respective probability distributions.

Proof. We first show that $r_\psi \leq \frac{1}{2}$ for all $|\psi\rangle \in H_d$, which we illustrate in the left side of Figure 2 for the case where $d = 3$. Suppose there exists $|\psi\rangle$ such that $r_\psi = \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$. From Corollary 1, for all $|\phi\rangle$ orthogonal to $|\psi\rangle$, we have $r_\phi \leq \frac{1}{2} - \varepsilon$. In dimension $d \geq 3$, there exist non-orthogonal states $|\phi\rangle$, $|\phi'\rangle$ such that $\langle\psi|\phi\rangle = \langle\psi|\phi'\rangle = 0$, and $|\phi\rangle \neq |\phi'\rangle$. Then $r_\phi + r_{\phi'} \leq 1 - 2\varepsilon$. If we choose $|\phi\rangle$, $|\phi'\rangle$ such that $1 - 2\varepsilon < ||\phi - \phi'|| < 1$, then from Lemma 4, we have that μ_ϕ and $\mu_{\phi'}$ have trivial overlap even though $\langle\phi|\phi'\rangle \neq 0$. This contradicts the theory being maximally nontrivial.

We now show that $r_\psi \geq \frac{1}{2}$ for all $|\psi\rangle \in H_d$, as illustrated in the center of Figure 2. Suppose there exists $|\psi\rangle$ such that $r_\psi = \frac{1}{2} - \varepsilon$ for some $\varepsilon > 0$. Since $r_{\psi'} \leq \frac{1}{2}$ for all $|\psi'\rangle \in H_d$, thus $r_\psi + r_{\psi'} \leq 1 - \varepsilon$. If we choose $|\psi\rangle$, $|\psi'\rangle$ such that $1 - \varepsilon < ||\psi - \psi'|| < 1$, then μ_ψ and $\mu_{\psi'}$ have trivial overlap from Lemma 4 even though $\langle\psi|\psi'\rangle \neq 0$. This again contradicts maximum nontriviality. □

This immediately implies the following:

Corollary 2. *In dimensions $d \geq 3$, a symmetric ψ-epistemic theory is maximally nontrivial if and only if for any state $|\psi\rangle$ and for all $\delta > 0$ the measure μ_ψ integrated over the following region is nonzero:*

$$\left\{\lambda : \frac{1}{2} \leq |\langle\psi|\lambda\rangle|^2 \leq \frac{1}{2} + \delta\right\} \tag{15}$$

Moreover, $\mathrm{Supp}(\mu_\psi)$ *has measure zero on the set of* λ *such that* $|\langle\psi|\lambda\rangle|^2 < \frac{1}{2}$.

Proof. By Lemma 5, for any state $|\psi\rangle$ we have $r_\psi = \frac{1}{2}$. By rewriting the distance between states in terms of their inner product, the corollary follows from the definition of r_ψ in Equation 14. □

In Lemma 5, we showed that the radius r_ψ of every state ψ in a maximally nontrivial symmetric theory is $\frac{1}{2}$. We now use this to show that a certain set of ontic states is deficient. Recall that we say a set S is deficient for measurement M if S is not in $\mathrm{Supp}(\mu_{\phi_i})$ for any $\phi_i \in M$.

Corollary 3. *Given any symmetric, maximally nontrivial* ψ*-epistemic theory in* $d \geq 3$*, for any measurement basis* $M = \{\phi_i\}_{i=1}^d$*, the region*

$$R_M = \left\{\lambda : |\langle\phi_i|\lambda\rangle|^2 < \frac{1}{2}, i = 1, \ldots, d\right\}$$

is deficient except on a set of measure zero. (Note that, by elementary geometry, R_M *has positive measure if and only if* $d \geq 3$*.)*

Proof. By Corollary 2, for all $i = 1, \ldots, d$, the set $\mathrm{Supp}(\phi_i)$ must have measure zero over the region R_M. However, Equation 2 implies that any $\lambda \in R_M$ must be in $\mathrm{Nonzero}(\xi_{i,M})$ for some i even if it is not in $\mathrm{Supp}(\phi_i)$. This means that R_M is deficient except possibly on a set of measure zero. □

In general, deficiency occurs in any theory in $d \geq 3$ even without the symmetry assumption, as proved by Harrigan and Rudolph [11] using the Kochen-Specker theorem [2]. In Corollary 3, we showed that symmetry implies a specific *type* of deficiency.

To show that no strongly symmetric, maximally nontrivial theory exists, we first prove two simple results for ψ-epistemic theories in general. These results will help us to derive a contradiction for strongly symmetric, maximally nontrivial theories.

Lemma 6. *Given any two orthogonal states* $|\phi\rangle$ *and* $|\psi\rangle$*, the set* $\mathrm{Supp}(\mu_\phi) \cap \mathrm{Nonzero}(\xi_{\psi,M})$ *has measure zero for all measurements* M *that contain* ψ.

Proof. Suppose to the contrary that $\mathrm{Supp}(\mu_\phi) \cap \mathrm{Nonzero}(\xi_{\psi,M})$ has positive measure for some measurement M containing ψ. Then by definition, if the state $|\phi\rangle$ is measured using M, the outcome corresponding to $|\psi\rangle$ is returned with nonzero probability. But since $|\langle\psi|\phi\rangle|^2 = 0$, this contradicts the Born rule (Equation (1)). □

Lemma 7. *For any* $\alpha \in \Lambda$*, let* $B_\varepsilon(\alpha) = \{\lambda : ||\lambda - \alpha|| < \varepsilon\}$ *be an* ε*-ball around* α*, for some* $\varepsilon > 0$*. Given a measurement basis* $M = \{\phi_i\}_{i=1}^d$*, there exists some* j *such that*

$$\int_{B_\varepsilon(\alpha)} \xi_{j,M}(\lambda)\, d\lambda > 0. \tag{16}$$

Proof. For any such $\alpha \in \Lambda$ the following holds,

$$\int_{B_\varepsilon(\alpha)} \sum_{i=1}^{d} \xi_{i,M}(\lambda)\, d\lambda = \int_{B_\varepsilon(\alpha)} 1\, d\lambda > 0.$$

This then implies that there exists some j such that

$$\int_{B_\varepsilon(\alpha)} \xi_{j,M}(\lambda)\, d\lambda > 0.$$

□

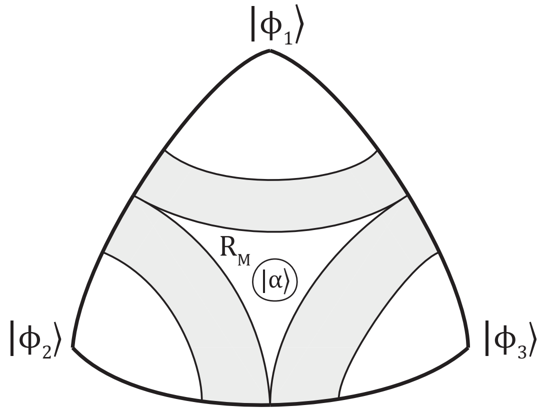

Figure 3: Pictorial representation of the deficiency region for $d = 3$. The shaded regions are the supports of the respective probability measures, and the middle unshaded region R_M is deficient.

Using these two results, we can now prove that in dimension $d \geq 3$, there exists no strongly symmetric, maximally nontrivial ψ-epistemic theory.

Theorem 1. *There exists no strongly symmetric, maximally nontrivial ψ-epistemic theory in dimension $d \geq 3$.*

Proof. Suppose we have a symmetric, maximally nontrivial theory in dimension $d \geq 3$, and we fix a measurement basis $M = \{\phi_i\}_{i=1}^d$. From Corollary 3, there exists a deficiency region given by

$$R_M = \left\{ \lambda : |\langle \phi_i | \lambda \rangle|^2 < \frac{1}{2}, i = 1, \ldots, d \right\},$$

perhaps minus a set of measure zero. This is illustrated in Figure 3 for the case where $d = 3$.

Consider $|\alpha\rangle = \frac{1}{\sqrt{d}} (|\phi_1\rangle + \cdots + |\phi_d\rangle)$, which is contained in the deficiency region. Given $\varepsilon > 0$, let $B_\varepsilon(\alpha) = \{\lambda : ||\lambda - \alpha|| < \varepsilon\}$ be the ε-ball around $|\alpha\rangle$. We choose ε such that $B_\varepsilon(\alpha)$ is contained in R_M. From Lemma 7, there exists some j such that $B := B_\varepsilon(\alpha) \cap \text{Nonzero}(\xi_{j,M})$ has nonzero measure. Without loss of generality, we assume that $j = 1$.

Let ν be the measure obtained by averaging μ_ψ over all states $|\psi\rangle$ orthogonal to $|\phi_1\rangle$, and let A be the set of all λ such that $|\langle \phi_1 | \lambda \rangle|^2 < \frac{1}{2}$. Since the theory is strongly symmetric, ν must be a function only of $|\langle \phi_1 | \lambda \rangle|^2$. Moreover, each of the measures μ_ψ assigns positive measure to the region of states λ such that $|\langle \psi | \lambda \rangle|^2$ is close to $\frac{1}{2}$, hence the averaged measure ν assigns positive measure to every open subset of A, and therefore in particular to B. This contradicts Lemma 6, which implies that each of the averaged measures μ_ψ must assign zero measure to $\text{Nonzero}(\xi_{1,M})$ and hence B. □

3.1 Proof of Generalized No-Go Theorem

We now generalize our proof of Theorem 1 to the "merely" symmetric case, where the probability distributions μ_ψ can vary with ψ. First note that our previous proof does not immediately carry over. Since the probability distributions can vary as ψ changes, it is possible that the distributions for states orthogonal to ϕ_1 might be able to "evade" the set B in the proof of Theorem 1 which returns answer ϕ_1 under measurement M, while maintaining some density near their outer radii.

To see how this might occur, consider the following one dimensional example: Let $\Lambda = \mathbb{R}$ be the real line. Construct $B \subseteq [0, 1]$ to be a "fat Cantor set" on $[0, 1]$ as follows. Initially set $B = [0, 1]$. In step 1, remove the middle 1/4 of this interval, so that $B = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$. At the i^{th} step, remove the middle $\frac{1}{2^{2i}}$ of each of the 2^i remaining intervals. Continue indefinitely. The resulting set B

is called a "fat Cantor set" because it is nowhere dense (so contains no intervals), yet has positive Lebesgue measure on $[0, 1]$.

For each point $x \in \mathbb{R}$, let μ_x be the uniform distribution on $[x-1, x+1]$ with B removed. Then μ_x is absolutely continuous with respect to the Lebesgue measure for all $x \in \mathbb{R}$, and furthermore has positive measure on $[1 + x - \varepsilon, 1 + x]$ for all $\varepsilon > 0$. However, despite the fact that B has positive measure, the distributions μ_x never intersect B. The worry is that our distributions in $\mathbb{CP}^{d-1}$ could likewise evade the set B in our proof, foiling our contradiction. This worry is related to a variant of the Kakeya/Besicovitch problem, as we discuss in Section 4.

We can extend Theorem 1 without solving a Kakeya-like problem, but to do so we will need a result about the differential geometry of $\mathbb{CP}^{d-1}$. Interestingly, we will use the fact that we are working in a complex Hilbert space; we believe the proof could be adapted to a real Hilbert space, but it would be much less convenient.

Discussing the differential geometry of $\mathbb{CP}^{d-1}$ is easiest if we first to pick a gauge for $\mathbb{CP}^{d-1}$, that is, if we pick a representative from each equivalence class of vectors which differ only by a global phase. We use the following gauge: let $|\alpha\rangle = \frac{1}{\sqrt{d}}(|\phi_1\rangle + \cdots + |\phi_d\rangle)$. For each equivalence class, we pick a representative u such that $\langle\alpha|u\rangle$ is real and positive. This uniquely identifies representatives for all equivalence classes of states, except those orthogonal to α. Moreover, this way of choosing a gauge is continuous and smooth near α; more precisely, equivalence classes which are close to one another have representatives which are also close to one another. This allows us to integrate over the manifold near α using these representatives. Using this gauge, we now prove the following.

Lemma 8. *Let M be a measurement basis $\{\phi_i\}$, let $|\alpha\rangle$ be defined as above, and let $d \geq 3$. Then there exist d vectors $u_1 \dots u_d$ in $\mathbb{CP}^{d-1}$ such that*

- $\langle u_i|\phi_i\rangle = 0$ *for all i.*
- $\langle u_i|\alpha\rangle = \frac{1}{\sqrt{2}}$ *for all i.*
- *The tangent vectors t_i to the geodesics from u_i to α are linearly independent at α when the tangent space is viewed as a real vector space.*

Proof. Let $a = \sqrt{\frac{d}{2(d-1)^2}}$ and $b = \sqrt{\frac{d-2}{4(d-1)}}$. Then we define $u_1, \dots u_{d-2}$ as follows:

$$u_i = \left(\sum_{j \neq i} a|\phi_j\rangle\right) + ib|\phi_{i+1}\rangle - ib|\phi_{i+2}\rangle\,,$$

For the last two vectors, we set

$$\begin{aligned} u_{d-1} &= \left(\sum_{j \neq d-1} a|\phi_j\rangle\right) + ib|\phi_d\rangle - ib|\phi_1\rangle\,, \\ u_d &= \left(\sum_{j \neq d} a|\phi_j\rangle\right) + b|\phi_1\rangle - b|\phi_2\rangle\,. \end{aligned}$$

Note that the coefficients in u_d are all real, unlike for the other $d-1$ vectors. It is straightforward to verify that $\langle u_i|\phi_i\rangle = 0$ and $\langle u_i|\alpha\rangle = \frac{1}{\sqrt{2}}$ for all i. Furthermore, we can compute the tangent vectors t_i as follows. The geodesics from $|u_i\rangle$ to $|\alpha\rangle$ in the Fubini-Study metric can be parameterized by

$$\gamma(t) = \cos(t)|v_i\rangle + \sin(t)|\alpha\rangle$$

where v_i is the normalized component of u_i orthogonal to α, that is $v_i = k\,(u_i - \langle\alpha|u_i\rangle\alpha)$ for some real normalization constant k. These geodesics lie entirely within our choice of gauge. Therefore t_i is the projection of $\gamma'(t)|_{t=\pi/2}$ onto the plane orthogonal to α, which is

$$t_i = u_i - \langle\alpha|u_i\rangle\alpha$$

Since t_i is in the tangent space, its normalization is irrelevant. Also, since our gauge is fixed, there is no ambiguity that u_i or t_i could be multiplied by a global phase.

We now verify that the t_i's are linearly independent. Suppose that $c_1t_1 + ... + c_dt_d = 0$, with c_i real. Note that $\langle\alpha|u_i\rangle\alpha$ has all real coefficients, so a coefficient of t_i is imaginary if and only if the corresponding coefficient of u_i is imaginary. Since $c_1t_1 + ... + c_dt_d = 0$, in particular the imaginary terms in $|\phi_i\rangle$ must sum to zero for all i. For $i = 3 \dots d$, only the terms $c_{i-2}t_{i-2}$ and $c_{i-1}t_{i-1}$ contain imaginary multiples of $|\phi_i\rangle$. Hence this constraint implies $c_{i-2} = c_{i-1}$. Additionally, c_1t_1 is the only term containing an imaginary multiple of $|\phi_2\rangle$, so we must have $c_1 = 0$. Therefore $c_1 = c_2 = \dots = c_{d-2} = 0$. Since $c_{d-1}t_{d-1}$ is the only term containing an imaginary multiple of $|\phi_1\rangle$, we must have $c_{d-1} = 0$, and hence $c_d = 0$ as well. Therefore the t_i's are linearly independent. □

Note that in a real Hilbert space, the analogous statement to Lemma 8 is false because the dimension of the tangent space at α is only $d-1$. In a complex Hilbert space the dimension of the tangent space is $2d-2$, so the tangent space can contain d linearly independent vectors assuming $d \geq 2$.

We now show that Lemma 8 implies the existence of a set B of positive measure, on which every μ_{u_i} is "equivalent" to the Lebesgue measure, in the sense that if $S \subseteq B$ has positive Lebesgue measure, then S has positive measure under each μ_{u_i}.

Lemma 9. *Let u_i and $|\alpha\rangle$ be as defined in Lemma 8. Then there exists a set B in the neighborhood of $|\alpha\rangle$, of positive Lebesgue measure, such that the μ_{u_i} are equivalent to the Lebesgue measure on B.*

Proof. Consider

$$B = N_\varepsilon(|\alpha\rangle) \cap \mathrm{Supp}(\mu_{u_1}) \cap \mathrm{Supp}(\mu_{u_2}) \cap \dots \cap \mathrm{Supp}(\mu_{u_d})$$

where $N_\varepsilon(|\alpha\rangle)$ denotes the ε-neighborhood of α. For sufficiently small ε, B can be shown to have the desired properties using the same techniques as the proof of Lemma 4. □

From these two lemmas, the proof of our main theorem follows, because the orthogonality of each u_i to ϕ_i and the Born rule implies the set B cannot give any outcome with positive probability under measurement, which contradicts the fact that each element in B must give some outcome under measurement.

Theorem 2. *There exists no symmetric, maximally nontrivial ψ-epistemic theory in dimension $d \geq 3$.*

Proof. By Lemmas 8 and 9 there is a measurement basis $M = \{\phi_1, \dots, \phi_d\}$ and vectors $u_1, \dots, u_d$ such that each u_i is orthogonal to ϕ_i.

Furthermore, there is a set B of positive measure such that each μ_{u_i} is equivalent to the Lebesgue measure on B. Therefore by the Born rule, for each i we must have

$$\int_B \mu_{u_i}(\lambda)\xi_{i,M}(\lambda)\mathrm{d}\lambda = 0$$

Since each μ_{u_i} is equivalent to the Lebesgue measure on B, this implies

$$\int_B \xi_{i,M}(\lambda)\mathrm{d}\lambda = 0$$

But also, since for each state λ, $\sigma_i \xi_{i,M}(\lambda) = 1$, we have that

$$\sum_i \int_B \xi_{i,M}(\lambda) \mathrm{d}\lambda = \int_B \mathrm{d}\lambda > 0$$

which is a contradiction. $\square$

4 Conclusions and Open Problems

In this paper, we gave a construction of a maximally nontrivial theory in arbitrary finite dimensions. However, the theory we constructed is not symmetric and is rather unnatural. We then proved that symmetric, maximally nontrivial ψ-epistemic theories do not exist in dimensions $d \geq 3$ (in contrast to the $d = 2$ case, where the Kochen-Specker theory provides an example). Our impossibility proof made heavy use of the symmetry assumption. As for the assumption $d \geq 3$, we used that in two places: firstly and most importantly, to get a nonempty deficiency region (in Corollary 3), and secondly, to prove that $r_\psi = \frac{1}{2}$ in Lemma 5.

It might be possible to relax our symmetry assumption and to consider different ontic spaces, since deficiency holds for any ψ-epistemic theory in $d \geq 3$ even without a symmetry assumption. It would be particularly interesting to consider ontic spaces Λ that are larger than $\mathbb{CP}^{d-1}$, but that are still acted on by the d-dimensional unitary group $U(d)$.

Also, in our proof, we did not use the specific form of the Born rule, only the fact that projection of $|\psi\rangle$ onto $|\phi\rangle$ must occur with probability 0 if $\langle\psi|\phi\rangle = 0$. Additional properties of the Born rule might place further constraints on ψ-epistemic theories.

Interestingly, trying to generalize the proof of Theorem 1 directly to obtain a proof of Theorem 2 gives rise to a variant of the Kakeya/Besicovitch problem. Recall that to prove Theorem 1, we showed that ontic states in a set B in the neighborhood of α returned value j under measurement, and yet the average measure of states orthogonal to j had nontrivial support on B. Now if the measures $\mu_\psi = f_\psi(|\langle\psi|\lambda\rangle|^2)$ vary with ψ, it remains open whether or not the measures of states orthogonal to j must have support on B, or if instead it is possible for them to "evade" B to avoid contradicting the Born rule.

Placing this problem in the plane rather than in $\mathbb{CP}^{d-1}$, we obtain a clean Kakeya-like problem as follows. Let S be a subset of $\mathbb{R}^2$ with the following property. For all $x \in \mathbb{R}^2$ and $\varepsilon > 0$, S contains a set of circles, centered at x, that has positive Lebesgue measure within the annulus $\{y : |y - x| \in [1-\varepsilon, 1]\}$. Can the complement of S have positive Lebesgue measure? This question has been discussed on MathOverflow [12] but remains open.

Here are some additional open problems.

- An obvious problem is whether symmetric and nontrivial (but not necessarily *maximally* nontrivial) theories exist in dimensions $d \geq 3$.
- How does the size of the deficiency region scale as the dimension d increases?
- In the maximally nontrivial theory we constructed, the overlap between any two non-orthogonal states $|\psi\rangle, |\phi\rangle$ is vanishingly small: like $(\varepsilon/d)^{O(d)}$ as a function of the dimension d and inner product $\varepsilon = |\langle\psi|\phi\rangle|$. Is it possible to construct a theory with substantially higher overlaps – say, $(\varepsilon/d)^{O(1)}$? (Note that if $d \geq 3$, then the result of Leifer and Maroney [5] says that the overlap cannot achieve its "maximum" value of ε^2.)
- Can we construct ψ-epistemic theories with the property that an ontic state λ, in the support of an ontic distribution μ_ψ, can *never* be used to recover the quantum state ψ uniquely? (This question was previously asked by Leifer and Maroney [5], as well as by A. Montina on MathOverflow [9].)

- What can be said about the case of infinite-dimensional Hilbert spaces?

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