Computation in a Topological Quantum Field Theory

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Abstract

This report investigates the computational power of the particle excitations of topological phases of matter, as modeled by a Topological Quantum Field Theory. We ask two basic questions: i) how does the computational power of these excitations change as a function of the genus of a fixed 2-dimensional space-time? and ii) independent of any particular space-time, what structural properties of a TQFT govern its computational power? When restricted to a space-time with space-like degrees of freedom represented by a surface of genus g, we answer the first question by observing a q^g-fold degeneracy in the ground state of the TQFT resulting from the presence of abelian anyons with exchange statistics a q-th root of unity. Such a resource is a topologically fault-tolerant quantum memory; however, braiding of the abelian anyons dos not afford the implementation of arbitrary unitary transforms. We address the second question by studying the algebraic model of anyons via Tensor Categories, and discuss what is known as of December 2015.

1 Computation in a Topological Quantum Field Theory

1.1 Computing with Anyons

A characteristic property of quantum particles is the indistinguishability of identical particles. All photons in the world share the same dynamical behaviour: for any system composed of multiple identical photons a permutation of the positions of any two photons cannot have any effect on the dynamics of the system as a whole. It follows that $\mathcal{S}_n$ the symmetric group on n elements acts on an ensemble of n identical particles as a symmetry of the composite system; more precisely, the system's wave function remains unchanged up to a total phase. In this way Unitarity of our physical theories shows the resulting transform of the wave function is an element of the group $U(1)$ i.e. transforms by scalar multiplication by a complex phase $e^{i\theta}$.

It is well-known that particle exchanges in three-dimensional space transform in precisely two flavours, each corresponding to one of two 1-dimensional irreducible representations of $\mathcal{S}_n$. If the particles are bosons (e.g. photons), then an exchange of two particles is represented by the identity symmetry: the wave function is invariant, and the particles obey Bose statistics. If the particles are fermions (e.g. electrons in a metal), then the action of exchange is represented on the wave function as a multiplication by -1 i.e. the wave function changes sign. Correspondingly, the particles obey Fermi statistics.

However, in a space-time restricted to two spatial dimensions we instead find a wealth of exotic particle statistics. The restricted topology of a surface imposes a topological change to the symmetry in exchanging two particles; the exchange symmetry group is no longer modeled by the symmetric group $\mathcal{S}_n$, but instead by the braid group $\mathcal{B}_n$ (See Appendix **A**), as depicted in Figure 1.1. Vertical displacement is in the time direction and the n particles are subjected to movement exchanging their positions from a_i to b_i. The braids are formed from the individual particle's world-lines.

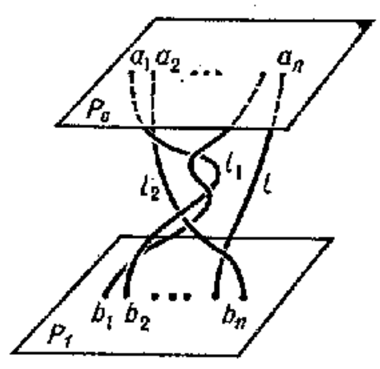


Figure 1: Particle world-lines tracing out a braid. From [14].

Contrary to $\mathcal{S}_n$, the infinite group $\mathcal{B}_n$ has infinitely many one-dimensional irreducible unitary representations, each corresponding to a choice of phase $e^{i\theta}$. We see that for $\theta = 0$ and $\theta = \pi$ we recover bosonic and fermionic particles, but for every other choice of θ we find a new particle. As these particles can obtain any change of phase upon permutation they have been dubbed **any**ons.

Furthermore, in certain topological quantum field theories (henceforth TQFTs) we find that the higher dimensional representations of B_n manifest themselves in an even more exotic type of particle; in contrast to those anyons that transform according to a one dimensional unitary representation i.e. according to a phase $e^{i\theta}$ in the abelian group $U(1)$, these **non-abelian** particles transform according to a representation V_d of $\mathcal{B}_n$ in some non-abelian unitary group $U(d)$ where $d > 1$ is the dimension of the corresponding representation. Fixing V_d, a pair of anyons of type V_d with movement confined to a surface trace out a topologically non-trivial braid formed by the particle's world-lines. This braid then corresponds via the representation to an element $U(d)$. These anyonic particles carry an intrinsic computational resource that is purely topological in nature. According to the unitary representation of $\mathcal{B}_n$ that they are modeled by, each particle carries internal degrees of freedom, and this state can be transformed by appropriately braiding the particles by moving them in 2-dimensional space-time. It is this computational resource of a TQFT that this report investigates.

Remark (On our treatment of TQFTs)**.** In order to not move too far afield, we focus on the strictly 2-dimensional part of a $2+1$-dimensional TQFT i.e. its anyonic excitations and corresponding braiding statistics. Furthermore, since a Unitary Modular Tensor Category (UMTC) in the sence of Tuarev corresponds uniquely to a TQFT, we will instead work with UMTCs. Appendices **B** and **C** contain an informal discussion of TQFT and UMTCs in general. For a complete treatment, see [12] and [13].

The rest of this report is divided into two sections: a review of the structure and classification of 2+1-dimensional TQFTs as it relates to the computational power of the TQFT, followed by a discussion of a fault-tolerant quantum memory in the topological degeneracy held by abelion anyons braided on a genus g surface. Here we find that changing the topology of the surface does not change the range of computations possible, but instead adds q^g degrees of freedom not accessible via local operations.

2 The Computational Power of a TQFT

2.1 Universality and Density of the Braid Group Representations

Given that each UMTC $\mathcal{C}$ (see Appendix **C**) models a topological state of matter, it follows that the computational power of the nonabelian statistics of an anyon native to a given $2+1$-dimensional TQFT modeled by $\mathcal{C}$ is governed by the algebraic structure of $\mathcal{C}$. Much like a choice of gate set in the standard Quantum Circuit Model may or may not afford efficient approximation of any quantum algorithm, the choice of UMTC governs the availability of a non-abelian anyon with braiding statistics supporting implementation of any quantum algorithm.

More precisely: given an object V in $\mathcal{C}$, the structure of a UMTC affords a homomorphism from the group algebra of the braid group

$$\mathbb{C}\mathcal{B}_n \to \mathrm{End}(V^{\otimes n})$$

acting on braid group generators σ_i (See Appendix **A**) as

$$\sigma_i \mapsto \mathrm{Id}_V^{\otimes i-1} \otimes \sigma_{V,V} \otimes \mathrm{Id}_V^{\otimes n-i-1}$$

where $\sigma_{V,V}$ denotes the braiding homomorphism

$$V \otimes V \to V \otimes V.$$

Furthermore, naturality of the braiding homomorphism implies compatability of braiding with the Hilbert Space structure on $\mathrm{End}(V^{\otimes n})$. It follows that the left action of $\mathrm{End}(V^{\otimes n})$ on itself induces a unitary representation

$$\mathcal{B}_n \to U(\mathrm{End}(V^{\otimes n})).$$

Thus for a given anyon type modeled by V, choosing $m, n \in \mathbb{N}$ anyons to encode one and two qubits respectively yields braid group representations of $\mathcal{B}_m$ and $\mathcal{B}_n$ in $\mathrm{End}(V^{\otimes m})$ and $\mathrm{End}(V^{\otimes n})$. If each of these representations have their images dense in the special unitary group, then we can apply the Solovay-Kitaev theorem to approximate any desired unitary to ϵ precision in a braid word of length $\mathrm{poly}(\log(\frac{1}{\epsilon}))$.

Example 2.1.

Freedman, Larsen and Wang characterize the images of the Jones representations of the Braid group in [7], deriving as corollary in [8] that the UMTC $\mathcal{C}(\mathfrak{sl}_2, e^{\pi i 5})$ consisting of highest weight representations obtained as a subquotient of the representation category of the Quantum Group $U_q\mathfrak{sl}_2$ specialized at $q = e^{\pi i 5}$ is universal. This particular UMTC corresponds to the SU(2)-Chern-Simons-Witten TQFT at level 3, or equivalently the Fibonacci UMTC $\mathcal{F}$ described in Appendix **B**.

Example 2.2. For an example lacking universality, by Jones in [10] the SU(2)-Chern-Simons-Witten TQFT at level 2 has braid group representations with image factoring through a finite group. In this case, the corresponding UMTC is $\mathcal{C}(\mathfrak{sl}_2, e^{\pi i 4})$.

2.2 Universality of a Given UMTC

Due to the divergence in computational power of different UMTCs, it is natural to ask for necessary and sufficient conditions for universality of the computational in a TQFT arising from a UMTC. As of December 2015, this remains an open problem. Recent progress by Etingof, Rowell and Wang have led to a conjectural characterization of UMTCs with braid group representations having exclusively finite image [5]:

Definition 1. *A UMTC $\mathcal{C}$ has property **F** if the associated representations of $\mathcal{B}_n$ on the centralizer algebras End$(V^{\otimes n})$ have finite image for all objects V and all $n \in \mathbb{Z}$.*

In any UMTC $\mathcal{C}$ we can associate to objects $X, Y \in \mathrm{Obj}(\mathcal{C})$ and a map $f : X \to Y$ a number called the **trace** of f denoted tr f. In particular, for the identity map $\mathrm{Id}_V : V \to V$ we can define the **dimension** $\dim(V) = \mathrm{tr}\ \mathrm{Id}_V$. For example, for any $V \in \mathcal{V}_k$, $\dim(X) \in \mathbb{N}$ is the standard vector space dimension. Denoting by $\dim(\mathcal{C}) = \sum_i \dim(V_i)^2$ the **global quantum dimension** of $\mathcal{C}$, then the conjecture, verified for all known examples, is

Conjecture 2.1. A UMTC $\mathcal{C}$ has property **F** if and only if $\dim(\mathcal{C}) \in \mathbb{N}$.

Returning to our standard example:

Example 2.3. Recall that the Fibonacci UMTC $\mathcal{F}$ was shown to be universal in [7]. Here we have two anyons 1 and τ, with dimensions

$$\dim(1) = 1 \text{ and } \dim(\tau) = \frac{1+\sqrt{5}}{2}.$$

It follows that

$$\dim(\mathcal{F}) = \frac{5+\sqrt{5}}{2},$$

in accord with $\mathcal{F}$ being universal and hence not having property **F**.

2.3 Classification of UMTCs of Low Rank

Recall the **rank** of a UMTC $\mathcal{C}$ is the number of simple objects in $\mathcal{C}$. In [4] Wang conjectured that UMTCs could be classified not only as representation categories of Quantum Groups, but also directly according to their rank. This conjecture was recently proven in [1] and is now called the **Rank Finiteness Theorem**:

Theorem 2.2. *There are finitely many isomorphism classes of UMTCs of fixed rank r.*

The theorem is directly analogous to a classical result of Landau's: for a fixed number $n \in \mathbb{N}$, there are only finitely many finite groups G with exactly n irreducible complex representations. The proof follows from analyzing the class equation

$$|G| = \sum_{i=1}^{n} [G : C(g_i)],$$

dividing by $|G|$ to yield the diophantine equation

$$1 = \sum_{i=1}^{n} \frac{1}{x_i}$$

which has finitely many solutions in the positive integers thus implying a bound on $|G|$; it follows there are only finitely many such G with n irreducible complex representations. Similarly, for a rank n UMTC $\mathcal{C}$, the proof of the rank finiteness theorem in [1] analyses the equation

$$\dim(\mathcal{C}) = \sum_{i=1}^{n} \dim(V_i)^2$$

in order to produce a bound on $\dim(\mathcal{C})$ implying a finite possible set of fusion rules for $\mathcal{C}$; a further analysis then reveals finitely many UMTCs having a fixed fusion rule, implying the result.

The Rank Finiteness Theorem suggests the feasability of a classification of UMTCs by rank. The process of classification can be understood from the axiomatic specification of a UMTC: each axiom imposes a polynomial constraint with $\mathbb{Z}$-coefficients, equating the classification of UMTCs with counting points on certain algebraic varieties (solution sets to polynomial equations). As of December 2015, UMTCs of rank 4 and lower have been classified via the galois groups associated to them, while in rank 5 all that is known is a list of possible fusion rules. The difficulty of the problem in rank 5 and greater can be described in terms of the higher dimensionality of the algebraic and arithmetic varieties involved [4].

According to the classification in [4], there are 70 UMTCs of rank less than or equal to 4, 10 of which are prime. The rest can be obtained from these 10 by applications of (categorical) direct sum and symmetry transformations. Table 1 below describes these, noting that 9/10 of the prime UMTCs are known to arise from a Quantum Group; here we label them according to the construction in [11]. We also explicitly name two UMTCs discussed in this report: **Fibonacci** and **Toric Code**. Column 3, row 4 is the only UMTC not known to arise as a coset construction from the category of representations of a Quantum Group; see [4] for details.

			[5] $Spin(16)$ **Toric Code** A
[1] Trivial A	[2] $SU(2)_1$ A	[2] $SU(3)_1$ A	[4] $SU(4)_1$ A
		[8] $E8_2$ NA	[4] $(G_2)_1 \times SU(2)_1$ [U]
	[2] $(G_2)_1$, also called **Fibonacci** F NA [U]	[2] $(A_1, 5)_{\frac{1}{2}}$ NA [U]	[2]$(G_2)_2$ NA [U]
			[3]$(G_2)_1 \times (G_2)_1$ NA [U]

Table 1: The ith column classifies rank i UMTCs. In each box we record: $[n]$ the number n of distinct theories in the isomorphism class; the lie group G and level k specifying the TQFT G_k by which the UMTC arises; whether the anyons are abelian (A) or non-abelian (NA); and $[U]$ the presence of a universal braiding anyon.

An example of a topological phase of matter is seen fractional quantum Hall liquids. These are electron systems confined to a disk and subjected to a perpendicular magnetic field at extremely low temperatures. Electrons in the disk, pictured classically as orbiting concentric annuli around the origin, organize themselves into a topological order. In this way the classification of UMTCs is akin to obtaining a periodic table of elements for the topological phases of matter.

3 Toric Code and Abelian Anyons

We now restrict our attention to abelion anyonic particle dynamics on a surface of genus g. To do this we first look at a model of qubits on a surface with genus $g = 1$ i.e. a torus. We then generalize this case to arbitrary genus.

3.1 Toric Code

The toric code, introduced by Kitaev, is an example of a symplectic or stabilizer code. Stabilizer codes are quantum analogues of classical linear codes. In general such a code works by selecting a set of "check operators", analogous to checksums, and encoding qubits in states belonging to the stabilizer space for each of the check operators. This allows for correction of a particular set of errors by first measuring a "syndrome" to determine the type of error—essentially checking to see if the state is still in the stabilizer space. By measuring the syndrome, and not the information in the bits directly, we can reconstruct the error and reverse it without damaging our state.

Consider an $r \times r$ lattice embedded on a torus. On each edge of the lattice we place a qubit (or a spin-$\frac{1}{2}$ degree of freedom). We then have two types of check operators. The first type we will call "star operators". These are operators

$$A_s^x = \prod_{j \in star(s)} \sigma_j^x$$

where $star(s)$ for some vertex s is the set of edges, or equivalently qubits, adjacent to that vertex. The second type we will call "face operators". These are defined by

$$A_u^z = \prod_{j \in boundary(u)} \sigma_j^z$$

where $boundary(u)$ is the set of qubits around a given face u. For the purposes of a stabilizer code, we now want to find what the stabilizer space of these operators looks like. We have $2r^2$ qubits, and $2r^2$ check operators. However, we have two relations among the check operators i.e.

$$\prod_s A_s^x = I = \prod_u A_u^z.$$

We conclude that the stabilizer space has dimension $2^2 = 4$.

Furthermore, each operator only acts on 4 qubits, and each qubit is only acted on by 4 operators. In addition, because the operators all commute (star operators and face operators commute because a boundary and a star either share 0 or 2 qubits) we can apply the syndrome measurement in a constant depth circuit. We can check that the errors undetectable by this code are only those errors that encompass an entire cycle i.e. an element of the homology group of the torus. We can show this either by working with the state space directly or by understanding the toric code in terms of anyonic particles.

3.2 Anyons and The Toric Code

Consider the same system as above—an $r \times r$ lattice with a qubit on each edge embedded on a torus. When we view this as a physical system we are led to the Hamiltonian

$$H = \sum_s (I - A_s^x) + \sum_u (I - A_u^z).$$

The null space of this operator is exactly the stabilizer code space of the toric code. Because the operator is nonnegative, we then see that the stabilizer space of the code is precisely the space of minimal energy for this Hamiltonian. It follows that the excited states, having non-minimal energy, then form the orthogonal complement.

Now suppose we have the ground state $|\xi\rangle$. We want to consider some minimal excited state $|\nu\rangle$, and to this end we can consider the excited states as ordered by the number of conditions they violate. For the state $|\xi\rangle$ we have that for all faces and vertices $A_s^x|\xi\rangle = A_u^z|\xi\rangle = |\xi\rangle$. Due to the relations described above, the number of violated conditions of a given type (either face or star) must be even. Let us consider the simplest case of two violated conditions at vertices s and p. Then we have at these vertices $A_s^x|\nu\rangle = -|\nu\rangle$ and the same for p. It is standard terminology to say that we have two **quasiparticles** at s and p. First, we note that we can generate the state $|\nu\rangle$ from the state $|\xi\rangle$. We simply apply a product of σ_j^x for all j along a path from s to p. We can also move quasiparticles along a path by applying products of σ_j^x along that path. Similarly, we can talk about quasiparticles on faces by considering violated conditions at faces. Again these can be created by products of σ_j^z for paths along the "dual lattice" i.e. the lattice given by thinking of faces as vertices, and connecting those vertices that correspond to adjacent faces.

We now consider what happens when these two types of particles interact. Suppose we have two vertex quasiparticles at s and p, and two face quasiparticles at u and ν. We consider the action of moving one of our face quasiparticles around a closed loop containing the vertex p. Moving this quasparticle around a closed loop is an application of star operators at every vertex inside the loop. All of these act as the identity except for the star operator at vertex p which flips the sign of our state. Moving one particle around another results in a phase change, even though the particles don't directly interact!

Finally, because we are working on a torus and not a plane, there is another action we can consider. Consider 4 operators, Z_1, Z_2, X_1, X_2, where the Z_1, Z_2 operators are products of σ^z around the two different nontrivial loops of the torus, and X_1, X_2 are products of σ_x along nontrivial loops. The commutation relations between these operators are

$$X_1X_2 = X_2X_1 \quad Z_1Z_2 = Z_2Z_1$$

$$X_1Z_1 = Z_1X_1 \quad X_2Z_2 = Z_2X_2$$

$$X_2Z_1 = -Z_1X_2 \quad X_1Z_2 = -Z_2X_1.$$

The operators X_1, Z_2 and Z_2, X_1 can each be thought of as acting as σ^x and σ^z operators on the two different qubits embedded by the toric code—in other words they act on the 4-dimensional space of ground states of the Hamiltonian. One way to think about this is to consider creating a pair of vertex and a pair of face particles. If we move one of the vertex particles around a nontrivial cycle, and then annihilate it with the other vertex particle, essentially performing X_1 we get a phase change, since we have also moved the particle in a closed path around a face particle. However, X_1 has to commute with the operators that create face particles, and so if we first perform X_1 and then create a pair of face particles, we should get an already-flipped state. In other words, the ground state "remembers" the topological behavior of past anyons.

3.3 Abelian Anyons on a Surface: A Generalization

What we have described above is simply a pair of particles such that when one is moved around the other the state vector acquires a phase flip of -1. It follows that these particles can be viewed as anyons with exchange statistics which are a fourth root of unity. Now consider a more general case: anyons on a torus with exchange statistics given by some $e^{i\theta}$ where $\theta = \pi\frac{p}{q}$. These are also known as rational anyons, since the phase change given by their exchange is a rational multiple of π. We will restrict our study to rational anyons.

Rather than looking at operators of particles moving around one another, we instead look at two operators C_1 and C_2, which are analogous to our X_i and Z_i operators above. The operator C_1 is given by the creation of a pair of anyons, moving one around one nontrivial cycle of the torus, and then fusing the anyons and annihilating them. The operator C_2 is the analogous operator for the other non-trivial loop. As above, these will act nontrivially on the ground state of the torus, and again such action is a topological behavior. We consider the commutator $[C_1, C_2] = C_2^{-1}C_1^{-1}C_2C_1$. Topologically what we have done is to move one particle in a loop around another, and this is equivalent to two exchanges. It follows that

$$[C_1, C_2] = e^{i2\theta}.$$

Consider some eigenvector $|\psi\rangle$ of C_1 with eigenvalue $e^{i\alpha}$ (since C_i's are unitary their eigenvalues have this form). Then we have the relation

$$[C_1, C_2]|\psi\rangle = e^{i2\theta}|\psi\rangle$$
$$C_2e^{i\alpha}|\psi\rangle = C_1C_2e^{i2\theta}|\psi\rangle.$$

Rearranging terms we can get

$$C_1(C_2|\psi\rangle) = e^{i(\alpha-2\theta)}(C_2|\psi\rangle).$$

In other words C_2 acts as a shift operator on eigenvalues of C_1. Furthermore we see

$$[C_1, C_2]^q = e^{qi2\pi\frac{p}{q}} = 1.$$

In summary, moving abelion anyons on a torus yields an additional q-fold degeneracy that does not exist when quasiparticles with the same exchange statistics move on a plane.

There is another way we can generalize this picture, and that is passing to surfaces of higher genus. Suppose then that the spatial degrees of freedom form a surface of genus g. The above analysis depends only upon the existence of 2 nontrivial loops on the torus. A genus g surface has $2g$ non-trivial loops. We organize these loops into pairs—one pair for each hole in our surface (a genus g surface can be thought of as a connected sum of g tori, or a surface with g holes). To each of these loops we associate an operator C_i^j where the $i = 1, 2$ and j ranges from 1 to g. Operators corresponding to loops in different pairs commute, and operators corresponding to the same loops commute. The only nontrivial commutator is $[C_1^j, C_2^j] = e^{i2\theta}$. We conclude then that on a surface of genus g, we have g copies of the q-fold degeneracy from the topology of the torus. In other words our ground state space has a q^g-fold degeneracy above and beyond what is present when abelians are braided on a genus 0 surface.

As the analysis shows, these abelian anyons on a genus g surface may have a finite dimensional topological degeneracy, but their braiding does not allow the implementation of arbitrary unitary transforms. At best we find a stable ground state protected from local errors.

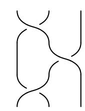
4 Conclusion

This report discussed the computational power of a topological quantum field theory. We reviewed two basic questions: (i) How does the toplogy of space-time affect the computational power of a TQFT and (ii) independent of a particular space-time, what intrinsic structural properties of a TQFT govern its computational power? The observation that abelian anyons on a genus g surface generalizes the toric code answers the first question: a non-trivial 2-D space-time topology provides an exponential (in the genus g) degeneracy in the ground state of the TQFT, a resource that in principle could serve as a quantum storage robust to local error, but cannot be used to implement any transformation not already accessible to anyons with comparable statistics on a genus $g = 0$ surface. To answer the second question, we appealed to the algebraic model of anyonic quantum computation via the Tensor Category formalism. By relying on our intuition from classical group theory to guide our understanding of their quantum generilizations, we have seen how the correspondence between UMTCs and 3-dimensional TQFTs, affords an algebraic and number theoretic analysis of the computational regime modeled by any TQFT. Recent deep results in this direction such as the **Rank Finiteness Theorem** and classification of UMTCs of low rank are the fruits of this approach: we are able to derive an explicit *periodic table of elements* for topological states of matter, indexed by the nature of the anyonic excitations provided.

Several open questions present themselves: we still do not have a good characterization of when a UMTC/TQFT has a universal computational model. Furthermore, restricting to those cases where the computational model is not universal, we do not have a good characterization of what computational power they do provide: at best we have a conjectural characterization of when arbitrary braiding of anyonic excitations implements at most a finite subgroup of the unitary group. Basic obstructions to answering these questions are a lack of understanding the relationships between the objects involved. For example, it's expected—but not yet known—that there are UMTCs that do not arise from Quantum Groups; in practice all known UMTCs can be constructed via some procedure from a Quantum Group. This could be rephrased as a gap in our understanding of the representation theory of a general Quantum Group. Given how basic computational questions directly translate into open problems concerning these relatively young fields, we expect new advances to directly contribute towards a deeper understanding of quantum computation arising from a TQFT in the years to come.

Appendix A The Braid Group

The braid group B_n on n strands, is a generalization of the symmetric group S_n on n symbols. It has an intuitive presentation as braids of n strings. Formally it is presented by $n-1$ generators σ_i which corresponds to crossing the i^{th} strand over the $i+1^{th}$ strand. For example, the braid



is given by generators

$$\sigma_1 \sigma_2 \sigma_1^{-1}.$$

Such a sequence of generators is called a **braid word**. This set of generators has a set of relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

where $|i-j| > 2$. Such relations are easy to see if you draw out the corresponding diagrams.

Appendix B: The Fibonacci Theory $\mathcal{F}$

We now introduce one of the simplest non-trivial TQFTs which we denote $\mathcal{F}$, known as the Fibonacci theory in the literature (See pp. 21 in [6]). The purpose of explicitly describing $\mathcal{F}$ here is to provide a concrete example to help motivate the abstract language of UMTCs used in this report.

We specify $\mathcal{F}$ by the structure of the anyonic excitations it supports in any 2 dimensional slice of space-time, as well as the resulting exchange statistics. In this model, there are only two different types of anyons, the vacuum (or absence of an anyon) denoted $\mathbf{1}$ and the non-abelian anyon τ. Part of the data specifying a TQFT is what is known as a **fusion rule**, which specifies what happens when two particles are brought together. Fusion can be considered as identifying the two anyons as a composite particle and identifying the resulting statistical behaviour of the ensemble. For example, fusing two fermions results in a boson. In the case of $\mathcal{F}$:

$$\tau \times \tau = \mathbf{1} + \tau,$$

$$\tau \times \mathbf{1} = \tau,$$

$$\mathbf{1} \times \tau = \tau,$$

$$\mathbf{1} \times \mathbf{1} = \mathbf{1}.$$

These rules should be read, for example in the first row, as stating that fusing two τ particles results in a superposition of two outcome states: the first state being the annihilation of both particles (the outcome being the vacuum) and the second state being the formation of another τ particle. The fact that this **fusion space** has dimension higher than 1 and so allows for superpositions is equivalent to τ being a non-abelian anyon, a fact that has only recently been proved; see [2].

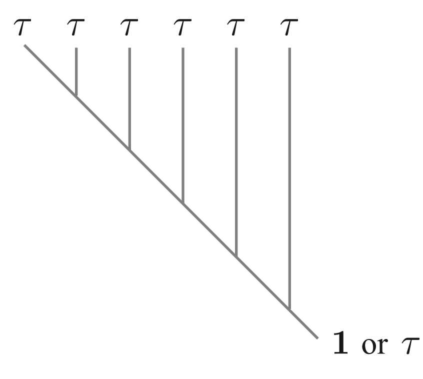


Figure 2: Fusion of multiple τ particles. From [15].

The computational power of $\mathcal{F}$ is now made apparent in the process of fusing n anyons of type τ. Consider a line of τ particles as depicted in figure 2 and proceed to fuse these particles in a step-wise fashion. We begin by fusing the first two particles, and then continue by fusing the outcome with the remaining particles incrementally. To each step i we assign an index e_i that indicates the outcome of the fusion at that step as being either $\mathbf{1}$ or τ. The states $|e_1, e_2, \ldots, e_{n-3}\rangle$ belong to what is called the fusion Hilbert space of the τ anyons, denoted $\mathcal{H}_n$. In principle, there are 2^{n-3} possible outcomes of e_i's, but not all are allowed by the fusion rules. For $n = 1$, we deal with the impossible case of the vacuum turning into a τ anyon, so

$$\dim(\mathcal{H}_1) = 0.$$

For $n = 2$ we see τ as input and output going through a trivial process so

$$\dim(\mathcal{H}_2) = 1.$$

Next, the possible outcomes are **1** or τ, giving

$$\dim(\mathcal{H}_3) = 1,$$

but at the very next step we see that there are two possible ways of yielding a τ from two different processes and so

$$\dim(\mathcal{H}_4) = 2.$$

Continuing this way we see that

$$\dim(\mathcal{H}_5) = 3,$$

and so on, yielding the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

which grows proportionally to ϕ^n where $\phi = \frac{1+\sqrt{5}}{2}$. Thus the fusion state space $\mathcal{H}_n$ of internal degrees of freedom of n anyons of type τ grows exponentially with n. In order for this to be a viable computational resource, we ideally want to be able to implement arbitrary unitary transforms on $\mathcal{H}_n$. As mentioned in the previous section, this is accomplished by braiding two τ anyons around one another as in Figure 2.3.

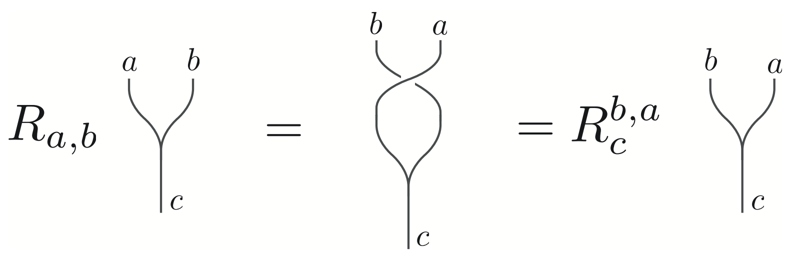


Figure 3: Braiding of two τ particles. From [15].

The process of doing so is a unitary transform on those basis vectors in the fusion space spanning possible fusion outcomes c according to the fusion rules governing a and b in the particular TQFT. Hence, this braiding matrix is part of what specifies a TQFT. In the case of $\mathcal{F}$ again, we find that exchanging any two τ anyons acts as the matrix

$$R_{\tau,\tau} = \begin{pmatrix} e^{\frac{4\pi i}{5}} & 0 \\ 0 & -e^{\frac{2\pi i}{5}} \end{pmatrix}.$$

Given an exponential state space, rules for braiding, and rules for fusing, we can now see that in order to encode a logical qubit as in the standard Quantum Circuit Model, we might employ four τ anyons. There are two distinguishable ways these anyons can be fused that can encode the qubit states: $|0\rangle = |\tau, \tau \to 1\rangle$ and $|1\rangle = |\tau, \tau \to \tau\rangle$. The possible logic gates are accomplished by forming arbitrary braids amongst these four qubits, with behaviour governed by $R_{\tau,\tau}$. Fusing and reading the outcome then correspond to a measurement in the QCM.

As our goal is to study the general computational power of a given TQFT, we find it fruitful to abstract the data presented here in the form of an algebraic model of anyons given by Tensor Categories as described in [12]. As described in Section 2, not only does the language of tensor categories simplify the description of a TQFT, but it also affords us algebraic and number-theoretic tools to analyse the computational power carried by any given TQFT.

Appendix C: TQFTs and Unitary Modular Tensor Categories

In the example furnished in Appendix B, a model of computation arose from a TQFT after specifying:

- a set of objects corresponding to the available types of anyons
- rules for fusing these objects pairwise

- rules for braiding these objects pairwise

We now capture this data in an algebraic structure known as a **Unitary Modular Tensor Category**. We include the concise definition for readers familiar with tensor categories, but for those not, immediately after we turn to recast the definition in terms of two familiar objects: i) the category $\mathcal{V}_k$ of finite dimensional vector spaces over the field k and ii) the TQFT $\mathcal{F}$.

Definition 2. *A **Unitary Modular Tensor Category** (abbreviated **UMTC**) is a semi-simple ribbon category $\mathcal{C}$ satisfying the following properties:*

- *$\mathcal{C}$ has only a finite number of isomorphism classes of simple objects.*
- *$\mathcal{C}$ is modular i.e. has non-degenerate S-matrix.*

We refer the interested reader to [12] for a detailed exposition of the terminology used here. Modularity, though important for the dictionary between TQFTs and UMTCs, will not play an immediate technical role in any of our subsequent discussions, so we omit covering it here. Instead, for our purposes we now quickly describe the structure of two UMTCs: $\mathcal{V}_k$ and $\mathcal{F}$.

$\mathcal{C}$ is a Semi-Simple Ribbon Category with finitely many isomorphism classes of simple objects

That $\mathcal{C}$ is a category means that $\mathcal{C}$ can be thought of as a collection of objects Obj($\mathcal{C}$). For example, in the category $\mathcal{V}_k$ of finite dimensional vector spaces over the field k, Obj($\mathcal{V}_k$) is the set of isomorphism classes of vector spaces over k of finite dimension; i.e. one object for each $n \in \mathbb{N}$. In the case of $\mathcal{F}$, = $\{\mathbf{1}, \tau\} \subset$ Obj($\mathcal{F}$), but the full collection of objects includes additional objects that can be constructed from these two, as soon shall be seen.

That $\mathcal{C}$ is a Semi-Simple Ribbon Category means that $\mathcal{C}$ has natural duals, natural tensor products, and a natural braiding structure on Obj($\mathcal{C}$). In the case of $\mathcal{V}_k$, we see for any $V, W \in$ Obj($\mathcal{V}_k$) these constructions are the familiar dual vector spaces:

$$V \to V^*$$

tensor product

$$V \otimes W$$

and braiding map coinciding with commutativity of tensor product

$$V \otimes W = W \otimes V.$$

In $\mathcal{F}$ the natural duals on Obj($\mathcal{F}$) are particle-to-antiparticle correlation

$$\mathbf{1} = \bar{\mathbf{1}}$$

and

$$\tau = \bar{\tau}$$

i.e. both $\mathbf{1}$ and τ are their own antiparticle (i.e. are self-dual). Tensor product then corresponds to fusion e.g.

$$\tau \times \tau \in \text{Obj}(\mathcal{F})$$

is the composite particle with behaviour governed by the collective statistical behaviour of two τ particles, and finally a natural braiding map given by

$$R_{\tau,\tau} : \tau \times \tau \to \tau \times \tau.$$

Finally, that $\mathcal{C}$ is semi-simple means each tensor product of objects can be decomposed into a direct sum of a distinguished class of **simple** objects. Here simple should be taken in the sense of an irreducible representation, prime number etc: a simple object is one that cannot be decomposed into a direct sum of smaller objects. For example, in $\mathcal{V}_k$, any vector spaces V and W of dimensions m and n decomposes as

$$V \otimes W = \bigoplus_{i=1}^{mn} k.$$

Analogously, the fusion rules of $\mathcal{F}$ specify the decomposition of the fusion of two τ particles:

$$\tau \times \tau = \mathbf{1} + \tau.$$

The simple objects of $\mathcal{V}_k$ clearly consist exclusively of the one dimensional vector space k, while the simple objects of $\mathcal{F}$ are $\{\mathbf{1}, \tau\}$.

Braid Group Representations and non-triviality of UMTCs

The preceeding section should have conferred the sense that UMTCs are in a sense categories that behave like the category of vector spaces, while at the same time UMTCs capture the basic data of anyonic statistics in a TQFT. While both of these form examples of a UMTC, they can be seen to lie on opposite ends of a vast spectrum of complexity in a UMTC: the former is in a certain sense trivial, while the latter is very interesting. To make this precise, consider the braiding maps in $\mathcal{V}_k$

$$V \otimes W \to W \otimes V$$

given by the familiar map $a \otimes b \to b \otimes a$, then this map squares to the identity. It follows that if in our TQFT we think of V and W as anyonic excitations, each with their worldlines (See Figure 4), then $V \otimes W$ would model the fused particle i.e. the composite subsystem consisting of the two particles. The braiding map then models the action on the fusion space when we

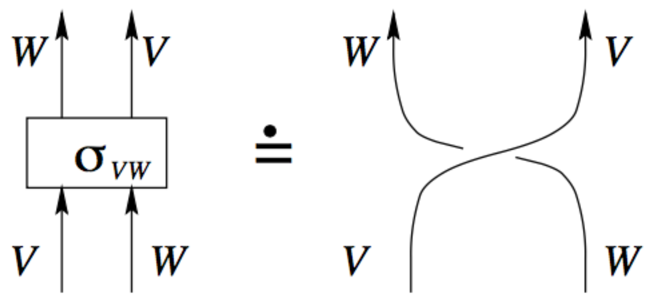


Figure 4: From [12]

twist one particle about the other; following this logic through, if we try to represent the braid group generator $\sigma_{V,W}$ that crosses the two strands corresponding to the worldlines of V and W (see appendix A), since this braid map squares to the identity, we have in effect limited ourselves to a highly constrained representation of the braid group $\mathcal{B}_2$. It follows that a UMTC such as $\mathcal{V}_k$ does not model any interesting anyonic exchange statistics. Conversely, $\mathcal{F}$ has braid map $R_{\tau,\tau}$ that yields a highly non-trivial braid group representation (as referenced in Section 2, this representation is computationally universal).

The above discussion illustrates that in order to find UMTCs modeling non-trivial TQFTs, we have to go outside the scope of classical algebra. One way to realize this principle is via the correspondence between certain TQFT and suitable categories of representations of a Quantum Group. For a survey of ways to obtain UMTCs in this way, see [9].

Unitarity, Modularity and TQFTs

Together with Unitarity which guarantees a given hermitian structure on $\mathrm{End}(V)$ for any object $V \in \mathcal{C}$, the Modularity condition (which we have not treated here) guarantees that to each UMTC we can uniquely associate a TQFT; the precise construction of a field theory from the abstract algebraic description of a tensor category is beyond the scope of this report. We instead refer the interested reader to now-standard reference [12].

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