On the complexity of stoquastic Hamiltonians

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Abstract

Stoquastic Hamiltonians, those for which all off-diagonal matrix elements in the standard basis are real and non-positive, are common in the physical world. We survey recent work on the complexity of stoquastic Hamiltonians. We discuss results relating stoquastic Hamiltonians and Merlin-Arthur games, including the result that stoquastic k-local Hamiltonian is StoqMA-complete, focusing in particular on the first non-trivial example of an MA-complete problem, stoquastic k-SAT.

1 Introduction

One of the central problems in physics and chemistry is that of finding the ground state energy and wave function of a quantum many-electron system. An important question, then, is how difficult it is to find the ground state energy of a given Hamiltonian. Kitaev first formalized this problem by introducing the class QMA, and showing that the problem of finding the ground state energy of local Hamiltonians is complete for QMA [1]. A more formal definition is below.

Definition 1.1. Given an n-qubit Hamiltonian $H = \sum_{i=1}^{m} h_i$ where each h_i is a Hermitian operator acting non-trivially on at most $k = O(1)$ qubits that satisfies $|h_i| \leq \mathsf{poly}(n)$ and $m = \mathsf{poly}(n)$, two constants a and b such that $0 \leq a < b$ with $b - a \geq 1/\mathsf{poly}(n)$, *$k$-local Hamiltonian* is the problem of determining whether

(YES) there exists an eigenvalue of H smaller than a, or
(NO) all eigenvalues of H are greater than b,

with the promise that one of these is true.

k-local Hamiltonian is a quantum generalization of k-SAT: both are quintessential complete problems for their respective classes; k-local Hamiltonian is QMA-complete for $k \geq 2$ just as k-SAT is NP-complete for $k \geq 3$. (Kempe *et al.* showed QMA-completeness of k-local Hamiltonian for $k \geq 2$ [2].) We can draw a stronger parallel with k-SAT by restricting to the case where all the local Hamiltonians h_i are k-local projectors.

Definition 1.2. *Quantum* k-SAT is k-local Hamiltonian with all the k-local Hamiltonians h_i restricted to be k-local projectors. In this case, the problem is of determining whether, given an n-qubit Hamiltonian $H = \sum_{i=1}^{m} \Pi_i$ where each Π_i is a k-local projection operator, $k = O(1)$, that satisfies $m = \mathsf{poly}(n)$, a constant b such that $b \geq 1/\mathsf{poly}(n)$,

$$\text{(YES) there exists a zero eigenvalue of } H\text{, or}$$
$$\text{(NO) all eigenvalues of } H \text{ are greater than } b,$$

with the promise that one of these is true.

Bravyi showed that quantum k-SAT is QMA_1-complete [3].

The two main results we review are complexity classifications of k-local Hamiltonian and quantum k-SAT for stoquastic Hamiltonians.

Definition 1.3. *Stoquastic* Hamiltonians are those Hamiltonians for which all off-diagonal matrix elements of the terms h_i in the standard basis are real and non-positive, that is,

$$\langle x|h_i|y\rangle \leq 0 \ \forall i \ \forall \ x, y \in \{0,1\}^n. \tag{1}$$

Stoquastic Hamiltonians arise naturally in the context of physics: the quantum transverse Ising model, the ferromagnetic Heisenberg model, the Jaynes-Cummings model of light-matter interaction, and most Hamiltonians reachable using Josephson-junction flux qubits (the kind used by D-Wave) are stoquastic.

Observe that if H is stoquastic, $\mathbb{1} - \beta H$ for β sufficiently small has only non-negative matrix elements. Thus we can apply the Perron-Frobenius theorem, setting the stage for many results involving stoquastic Hamiltonians.

Theorem 1.4. (Perron-Frobenius theorem) *If all elements of a real symmetric square matrix A are non-negative, then A has a largest real eigenvalue; furthermore, the components of the corresponding eigenvector can be chosen to be all non-negative.*

The largest eigenvalue of $\mathbb{1} - \beta H$ corresponds to the ground state energy of H. The eigenvectors in both cases are the same, so the ground state of a stoquastic Hamiltonian can be chosen to have non-negative amplitudes!

Finally, let us define the two major problems of interest in the context of stoquastic Hamiltonians.

Definition 1.5. *Stoquastic k-local Hamiltonian* is k-local Hamiltonian with a stoquastic Hamiltonian. Similarly, *stoquastic* k-SAT is quantum k-SAT with a stoquastic Hamiltonian.

At this point, it is not clear how much the restriction that all off-diagonal elements in the standard basis be non-positive might reduce the complexity of local Hamiltonian or quantum k-SAT (if at all), what stoquastic Hamiltonian problems are still QMA-complete (if stoquastic local Hamiltonian is not), and what insights stoquastic Hamiltonians might offer to complexity theory.

In this survey, we will answer these questions and build insight and intuition about what makes stoquastic local Hamiltonians 'simpler' than local Hamiltonians. We will follow a different order than progress in the field. In Section 2, we discuss the complexity of stoquastic local Hamiltonian, and mention a natural QMA-complete problem in the context of stoquastic Hamiltonians. In Section 3, we prove that stoquastic 6-SAT is MA-complete (indeed, the first such problem). Our proof leads into a brief discussion of those stoquastic Hamiltonians for which adiabatic evolution can be efficiently simulated.

The main focus of this paper is on the result that stoquastic 6-SAT is MA-complete: all proofs related to this result are included. Some other definitions and proofs are included in appendices. Appendix A includes a 'quantum' definition of MA and a matching definition of StoqMA as well as a discussion of the place of StoqMA in the framework of Merlin-Arthur games. Appendix B contains a sketch of the proof that stoquastic local Hamiltonian is StoqMA-complete.

2 Stoquastic local Hamiltonian

2.1 Stoquastic local Hamiltonian is MA-hard

The first question one might ask is how much k-local Hamiltonian is simplified by the condition that the Hamiltonian be stoquastic. Without the stoquastic requirement, k-local Hamiltonian was shown by Kitaev to be QMA-complete [1] using a clock construction.

We can show that stoquastic local Hamiltonian is MA-hard for $k \geq 6$ using a near-identical clock construction. However, instead of using a universal set of quantum gates, we use a classical reversible gate set that is universal (e.g. Toffoli gates). This is why we can only show that stoquastic local Hamiltonian is MA-hard for $k \geq 6$: we need 6-local terms in the clock construction, 3 for the Toffoli gates and 3 for the clock.

Let us write the Hamiltonian using the quantum definition of MA in Section A.1. Specifically, we use the Hamiltonian $H = H_{\text{in}} + H_{\text{prop}} + H_{\text{out}}$ acting on r qubits in the state $|+\rangle$ ('coin' qubits), K ancilla qubits, and an s-qubit witness state $|\psi\rangle$. Let V have L reversible classical gates R_ℓ, so we will use L qubits for the clock. Then with

$$\begin{aligned}
H_{\text{in}} &= \left(\sum_{i=1}^{r} \Pi^{(-)}_{\text{coin},i} + \sum_{i=1}^{k} \Pi^{(1)}_{\text{anc},i}\right) \otimes \Pi^{(0)}_{\ell=1}, \\
H_{\text{out}} &= \Pi^{(0)}_{q_{\text{out}}} \otimes \Pi^{(1)}_{\ell=L}, \text{ and} \\
H_{\text{prop}} &= \sum_{\ell=1}^{L} H_{\text{prop}}(\ell), \text{ where}
\end{aligned}$$

$$H_{\text{prop}}(\ell) = \begin{cases} \left(\Pi^{(0)}_{\ell}\Pi^{(0)}_{\ell+1} + \Pi^{(1)}_{\ell}\Pi^{(0)}_{\ell+1}\right) - R_\ell \otimes \left(|1\rangle\langle 0|_\ell + |0\rangle\langle 1|_\ell\right)\Pi^{(0)}_{\ell+1}, & \ell = 1 \\ \Pi^{(1)}_{\ell-1}\Pi^{(0)}_{\ell+1} + \Pi^{(0)}_{\ell-1}\Pi^{(1)}_{\ell} - R_\ell \otimes \Pi^{(1)}_{\ell-1}\left(|1\rangle\langle 0|_\ell + |0\rangle\langle 1|_\ell\right)\Pi^{(0)}_{\ell+1}, & 1 < \ell < L \\ \left(\Pi^{(1)}_{t-1}\Pi^{(0)}_{t} + \Pi^{(0)}_{t-1}\Pi^{(1)}_{t} + \Pi^{(1)}_{t-1}\Pi^{(1)}_{t}\right) - R_t \otimes \Pi^{(1)}_{t-1}\left(|1\rangle\langle 0|_t + |0\rangle\langle 1|_t\right), & \ell = L \end{cases} \tag{2}$$

we can use Kitaev's result [1] that if $\exists\,|\psi\rangle$ such that $P(1) \geq 1-\varepsilon$, then there exists an eigenvalue of H that is $\leq \varepsilon$, and if $\forall\,|\psi\rangle$, $P(1) \leq \varepsilon$, then all eigenvalues of H are $\geq c(1-\sqrt{\varepsilon})L^{-3}$ for some constant c. Noting that all the off-diagonal terms are non-positive—the only off-diagonal terms in the standard basis are the coin projectors and the Toffoli gates R_ℓ—we can reduce any problem in MA to a stoquastic 6-local Hamiltonian H.

2.2 Stoquastic local Hamiltonian is StoqMA-complete

In the previous section we used Kitaev's clock construction to show MA-hardness for stoquastic k-local Hamiltonian for $k \geq 6$. A natural next question is whether stoquastic k-local Hamiltonian is complete for any class. Indeed, it can be shown that stoquastic k-local Hamiltonian is StoqMA-complete (StoqMA was originally defined in [4]; a definition is included in Appendix A.2 for completeness). A sketch of the proof is included in Appendix B.

Interestingly, though determining the ground state energy of a stoquastic local Hamiltonian is only StoqMA-complete, finding the *highest* energy for a stoquastic Hamiltonian is QMA-complete. Jordan *et al.* show this in [5] by proving that finding the lowest eigenvalue of a symmetric stochastic matrix is QMA-complete (if S is a symmetric stochastic matrix, then $-S$ is stoquastic).

3 The first MA-complete problem

3.1 Stoquastic k-SAT

The two great successes of the study of quantum computing are its contributions to our understanding of complexity theory and to our understanding of the physical world. In this section we will review the "discovery" of the first non-trivial MA-complete (promise) problem [4]. Until this discovery, no non-trivial complete problems were known for either BPP or MA; even now, no non-trivial complete problems for BPP are known (unless P=BPP). This first non-trivial MA-complete problem is stoquastic 6-SAT. We will discuss the proof that it is MA-complete in two stages, first showing MA-hardness by reduction from stoquastic 6-local Hamiltonian, and then showing that stoquastic 6-SAT $\in$ MA by explicitly constructing Arthur's BPP algorithm. Surprisingly, the development of this algorithm additionally gives us insight into which quantum Hamiltonians can be efficiently simulated classically.

3.2 Stoquastic 6-SAT is MA-hard

We begin by showing that stoquastic 6-SAT is hard for MA. Consider stoquastic k-local Hamiltonian for $k \geq 6$. As discussed earlier, and as shown in Lemma 3 of [6], stoquastic k-local Hamiltonian is MA-hard for $k \geq 6$; in particular, it is MA-hard for $k = 6$.

How can we solve stoquastic k-local Hamiltonian instances with stoquastic k-SAT? If we replace each local Hamiltonian h_i with a projector Π_i projecting onto the ground space of h_i, each projector Π_i has non-negative matrix elements because the corresponding local

Hamiltonian h_i has non-positive off-diagonal elements. Furthermore, for the YES case, any satisfying state of the stoquastic k-SAT problem is the ground state of the corresponding stoquastic k-local Hamiltonian, and for the NO case, the best non-satisfying state of the stoquastic k-SAT problem has penalty at least ε (by the promise) and so is a NO case of k-local Hamiltonian. Thus stoquastic k-SAT is MA-hard for $k \geq 6$.

3.3 Stoquastic 6-SAT is in MA

We will show that stoquastic k-SAT is in MA for any positive constant k by explicitly giving Arthur's BPP algorithm. Given an instance of stoquastic k-SAT with the projectors $\{\Pi_i\}_{i=1}^m$, define the Hermitian operator

$$G = \frac{1}{m} \sum_{i=1}^{m} \Pi_i. \tag{3}$$

Note that the matrix elements of G are non-negative in the standard basis. For a YES case, the largest eigenvalue of G is 1 (in which case a satisfying instance $|\theta\rangle$ exists and is in the ground space of all the projectors, so $G|\theta\rangle = \frac{1}{m}\sum_i \Pi_i |\theta\rangle = |\theta\rangle$). For a NO case, the largest eigenvalue of G is less than or equal to $1 - \varepsilon/m$ (at least one constraint is violated).

Arthur will distinguish between these two cases using a random walk. Merlin provides the starting point, and Arthur performs the walk, as well as a series of checks at each step which are passed for a YES case. Let us consider in more detail how the walk works for a YES case.

Suppose that there exists a satisfying assignment $|\theta\rangle = \sum_{x\in\{0,1\}^n} \theta_x |x\rangle$. By the Perron-Frobenius theorem, this satisfying assignment can be chosen with $\theta_x \geq 0\ \forall x$. (We are finding the ground state of the stoquastic Hamiltonian $\mathbb{1} - \sum_i(\beta\Pi_i)$, where $\beta^{-1} = \sum_i |h_i|$ ensures stoquasticity.) Merlin provides Arthur with the starting point of the walk, $x_0 = \arg\max_x \theta_x \in \{0,1\}^n$. The steps Arthur takes to verify that there exists a satisfying assignment, given this initial x, are as follows.

For $i \in [0, L]$:

1. Arthur verifies that $\langle x_i| \Pi_j |x_i\rangle > 0\ \forall j$.
2. Arthur finds the set $\{y \in \{0,1\}^n : \langle x_i| G |y\rangle > 0\}$. This set has at most $2^k m = \mathsf{poly}(n)$ elements.
3. For each y, Arthur chooses the first j such that $\langle x_i| \Pi_j |y\rangle > 0$, and computes the transition probability
$$P_{x_i \to y} = \langle x_i| G |y\rangle \sqrt{\frac{\langle y| \Pi_j |y\rangle}{\langle x_i| \Pi_j |x_i\rangle}}. \tag{4}$$
4. Arthur checks that he actually has a random walk, i.e. that $\sum_y P_{x_i \to y} = 1$.

5. Arthur randomly chooses the next step in the walk, x_{i+1}, according to the transition probabilities. He stores

$$r_i = \frac{P_{x_i \to x_{i+1}}}{\langle x_i| G |x_{i+1}\rangle} = \sqrt{\frac{\langle x_{i+1}| \Pi_j |x_{i+1}\rangle}{\langle x_i| \Pi_j |x_i\rangle}}. \tag{5}$$

Finally, Arthur checks that $\prod_{i=0}^{L-1} r_j \leq 1$. ($r_L$ is unused.)

In general, the probability of passing the checks in Steps 1 and 4 depends on the largest eigenvalue of G. Note that the seemingly strange square root term $\sqrt{\frac{\langle y|\Pi_j|y\rangle}{\langle x|\Pi_j|x\rangle}}$ is in fact just θ_y/θ_x. This is obvious when $\mathrm{Rank}(\Pi_j) = 1$, and generalizing to $\mathrm{Rank}(\Pi_j) > 1$ is not difficult. The condition $\langle x| \Pi_j |y\rangle > 0$ ensures that $|x\rangle$ and $|y\rangle$ are in the same block of Π_j so that this holds.

In a YES case, we succeed with probability 1. Arthur has a proper random walk because $\sum_y P_{x_i \to y} = \sum_y \langle x| G |y\rangle \theta_y/\theta_x = \sum_y \langle x| G |y\rangle \langle y| |\theta\rangle /\theta_x = \langle x| G |\theta\rangle /\theta_x = 1$. More details can be found in [4]. For a NO case, the success probability is upper-bounded by the highest eigenvalue of G, which is $\leq 1 - \varepsilon/m$. The success probability is $P(1) = \sum_{\{x_i\}_{i=0}^{L} \subseteq S_{\text{good}}} \left(\prod_{i=0}^{L-1} P_{x_i \to x_{i+1}}\right) \leq \sum_{\{x_i\}_{i=0}^{L} \subseteq S_{\text{good}}} \left(\prod_{i=0}^{L-1} \langle x_i| G |x_{i+1}\rangle\right)$ (by the r_j product check). The term in the product is necessarily smaller than the largest eigenvalue of G, so $P(1) \leq 2^{-n/2}(1 - \varepsilon/m)^L$ for a NO case. $\varepsilon \geq 1/\mathsf{poly}(n)$, $m = \mathsf{poly}(n)$, so we can make this smaller than 1/3 with only $L \leq \mathsf{poly}(n)$ steps in the random walk.

It is interesting to note that the random walk works for simulating evolution or drawing from the state $|\theta(t)\rangle$ if the Hamiltonian is a YES case of stoquastic k-SAT, that is, if it is "frustration-free". This notion is explored more thoroughly in [7]. Research in this area continues: last year Bravyi and Hastings showed that quantum annealing with the transverse-field Ising model is equivalent to quantum annealing with a class of stoquastic Hamiltonians that includes the current D-Wave machine Hamiltonian, that is, Hamiltonians which can be written as a sum of a k-local diagonal Hamiltonian and a 2-local stoquastic Hamiltonian [8].

We have thus shown that stoquastic k-SAT is MA-hard for $k \geq 6$ and in MA for any positive k. Stoquastic 6-SAT is the smallest k for which the problem is known to be MA-complete and indeed is the first non-trivial MA-complete problem.

As discussed in Appendix A.2, StoqMA $\subseteq$ AM, so a proof of a separation between MA and StoqMA (for example, by showing that stoquastic k-local Hamiltonian is more difficult than stoquastic k-SAT) would also show NP $\not\subseteq$ P/poly. For reference, Figure 1 is an inclusion diagram of the complexity classes discussed.

4 Conclusions

We have surveyed the complexity of stoquastic Hamiltonians, building up to a proof that stoquastic k-SAT is MA-complete, and indeed, is the first such non-trivial problem. We have shown how the added requirement of stoquasticity of the Hamiltonian simplifies k-local Hamiltonian from QMA-complete to StoqMA-complete, and how stoquasticity simplifies quantum k-SAT from QMA_1-complete to MA-complete. The introduction of StoqMA leads to another possibility for separating MA and AM, by separating MA and StoqMA which seem more comparable. We additionally discussed a natural QMA-complete problem for stoquastic local Hamiltonians, as well as work on determining which Hamiltonians can be efficiently classically simulated.

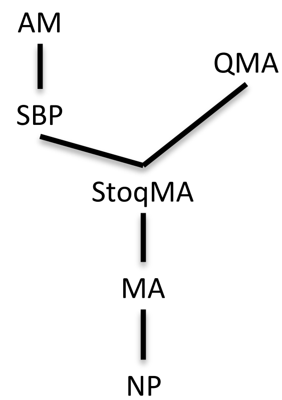

Figure 1: Inclusion diagram of the complexity classes discussed.

References

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A Complexity classes: MA and StoqMA

In this section we will introduce a 'quantum' definition of MA and a (also 'quantum') definition of StoqMA.

A.1 MA

We begin with a quantum definition of MA ("MA_q"), which Bravyi *et al.* show to be equivalent to the usual definition [6]. The quantum definition of MA simplifies the proof that the stoquastic 6-SAT is MA-hard. The verifier takes as input a *classical* state $|\psi\rangle$ from Merlin and two sets of ancilla qubits, one with all qubits in the state $|+\rangle$ to simulate randomness (the 'coin' qubits), and one with all qubits in the state $|0\rangle$. Arthur ultimately only measures a single qubit in the z basis; in the YES case, he measures 1 with high probability, and in the NO case he measures 0 with high probability. In essence, Merlin sends a witness state which Arthur verifies with a BPP circuit.

Definition A.1. A problem belongs to the class MA_q if and only if there exists a uniform family of polynomial-time quantum verifier circuits V_x with X, CNOT, and Toffoli gates such that for classical input x,

$$\text{(YES)}\ \exists\, |\psi\rangle \in (\mathbb{C}^2)^{\otimes p(|x|)} \text{ such that } P[V(|x\rangle\,|\psi\rangle\,|+\rangle^{\otimes r}\,|0\rangle^{\otimes K} = 1] \geq 2/3, \text{ and}$$
$$\text{(NO)}\ \forall\, |\psi\rangle \in (\mathbb{C}^2)^{\otimes p(|x|)},\ P[V(|x\rangle\,|\psi\rangle\,|+\rangle^{\otimes r}\,|0\rangle^{\otimes K}) = 1] \leq 1/3$$

where p is some polynomial, and r and $K \in \mathsf{poly}(|x|)$ are the number of 'coin' and ancilla qubits.

How can we see that this is equivalent to the regular definition of MA? $\mathsf{MA}_q \subseteq \mathsf{MA}$ because $P(1) = \langle\psi|\, M\, |\psi\rangle$ where $M = \langle + + \ldots +|\, \langle 00\ldots 0|\, V^\dagger \Pi_1^{(1)} V\, |+ + \ldots +\rangle\, |00\ldots 0\rangle$ is diagonal in the standard basis, with $\Pi_1^{(1)}$ the projector for the first qubit onto the 1 state. To show $\mathsf{MA} \subseteq \mathsf{MA}_q$, note for YES cases that the MA classical witness can be used for MA_q. For NO cases, Merlin cannot cheat by sending a quantum witness because $P(1) = \langle z|\, M\, |z\rangle \leq 1/3\ \forall z \in \{0,1\}^*$ and M is diagonal in the standard basis; all eigenvalues of M are thus $\leq 1/3$.

A.2 StoqMA

There are two significant differences between MA and StoqMA. First, the final measurement for StoqMA is non-classical: the final qubit measurement is $|+\rangle$ rather than $|1\rangle$, and the gap between the threshold probabilities has polynomial rather than constant separation. Formally,

Definition A.2. A problem belongs to the class StoqMA if and only if there exists a uniform family of polynomial-time quantum verifier circuits V_x with X, CNOT, and Toffoli gates such that for classical input x,

$$\text{(YES)}\ \exists \left|\psi\right\rangle \in (\mathbb{C}^2)^{\otimes p(|x|)} \text{ such that } P[V(\left|x\right\rangle \left|\psi\right\rangle \left|+\right\rangle^{\otimes r} \left|0\right\rangle^{\otimes K} = +] \geq \epsilon_{\text{yes}}, \text{ and}$$
$$\text{(NO)}\ \forall \left|\psi\right\rangle \in (\mathbb{C}^2)^{\otimes p(|x|)},\ P[V(\left|x\right\rangle \left|\psi\right\rangle \left|+\right\rangle^{\otimes r} \left|0\right\rangle^{\otimes K}) = +] \leq \epsilon_{\text{no}},$$

where p is some polynomial, r and $K \in \mathsf{poly}(|x|)$ are the number of 'coin' and ancilla qubits, and $0 \leq \epsilon_{\text{no}} < \epsilon_{\text{yes}} \leq 1$ with $\epsilon_{\text{yes}} - \epsilon_{\text{no}} \geq 1/\mathsf{poly}(n)$.

In contrast to other probabilistic classes like BPP, MA, and QMA, for StoqMA amplification of the gap between YES and NO by repeated measurement and majority vote is impossible. We can show that $\mathsf{StoqMA} \subseteq \mathsf{SBP}$ by showing that stoquastic k-local Hamiltonian is contained in SBP. Because $\mathsf{SBP} \subseteq \mathsf{AM}$, this shows that $\mathsf{StoqMA} \subseteq \mathsf{AM}$.

B Proof sketch that stoquastic local Hamiltonian is StoqMA-complete

The idea behind the proof is to find constants $\alpha > 0$, $\beta \in \mathbb{R}$ such that

$$P\left[V(\left|x\right\rangle \left|\psi\right\rangle \left|+\right\rangle^{\otimes r} \left|0\right\rangle^{\otimes K} = +\right] = \left\langle\psi\right| (-\alpha H + \beta \mathbb{1}) \left|\psi\right\rangle \forall \left|\psi\right\rangle \in (\mathbb{C}^2)^{\otimes n}, \tag{6}$$

where x is a classical description of H. But how can we convert H into observables proportional to $\left|+\right\rangle\left\langle+\right|$? Bravyi *et al.* prove this with two lemmas [4]: first, by showing that for any k-local Hamiltonian H, there exist real constants $\alpha > 0$ and β such that a decomposition

$$\alpha H + \beta \mathbb{1} = \sum_i p_i U_i H_i U_i^\dagger, \tag{7}$$

can be found efficiently, where $\sum_i p_i = 1$ and U_i is an n-qubit circuit using only X and CNOT gates, and each H_i is either $-\left|0\right\rangle\left\langle0\right|^{\otimes k}$ or $-X \otimes \left|0\right\rangle\left\langle0\right|^{\otimes k-1}$. At this stage, both measurements can be reduced by a 'stoquastic isometry' W_i to measurement of X only. Then we have

$$\alpha H + \beta \mathbb{1} = -\sum_i p_i W_i X W_i^\dagger, \tag{8}$$

where X acts on the qubit to be measured, and $\{W_i\}$ is a family of stoquastic isometries mapping k to $2k+1$ or $2k-1$ qubits (corresponding respectively to the terms $-\left|0\right\rangle\left\langle0\right|^{\otimes k}$ and $-X \otimes \left|0\right\rangle\left\langle0\right|^{\otimes k-1}$). Thus, for each term we can construct a verifier V_i and use it for the measurement. Using the fact that $X = 2\left|+\right\rangle\left\langle+\right| - \mathbb{1}$, we find

$$\sum_i p_i P\left[V_i(\left|x\right\rangle \left|\psi\right\rangle \left|+\right\rangle^{\otimes r} \left|0\right\rangle^{\otimes K} = +\right] = 1 - \frac{\left\langle\psi\right| (\alpha H + \beta \mathbb{1}) \left|\psi\right\rangle}{2}. \tag{9}$$

The final thing to note is that stoquastic verifiers form a convex set: this allows us to convert the sum over verifiers V_i into a single verifier V. This shows that stoquastic k-local Hamiltonian is in StoqMA.

Stoquastic k-local Hamiltonian can be shown to be hard for StoqMA using a perturbative approach. For any stoquastic verifier with L gates and for any precision parameter $\delta \ll L^{-3}$ (the perturbative expansion is in δ), one can define a stoquastic 6-local Hamiltonian whose smallest eigenvalue is

$$\lambda = \frac{\delta}{L+1}\left(1 - \max_{\psi} P\left[V(|x\rangle\,|\psi\rangle\,|+\rangle^{\otimes r}\,|0\rangle^{\otimes K} = +\right]\right) + O(\delta^2). \tag{10}$$

Up to first order in δ, then, we have $\lambda \leq \delta(1-\epsilon_{\text{yes}})/(L+1)$ for YES instances and $\lambda \geq \delta(1-\epsilon_{\text{no}})/(L+1)$ for NO instances. $\epsilon_{\text{yes}} - \epsilon_{\text{no}} \geq 1/\mathsf{poly}(n)$ implies the same separation between the ground state energies of YES and NO instances. Thus, stoquastic k-local Hamiltonian is StoqMA-complete.